ELEMENTS

OF

ALGEBRA.

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ALGEBRA.

BY A. F. LACROIX.

Fourteenth Edition.

TRANSLATED BY

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WITH NUMEROUS EXPLANATIONS AND AUGMENTATIONS OF EXAMPLES.

"l'Algèbre est une science de démonstration & ne doit être dégradée en un livre de règles."

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CONTENTS.

ERRATA.

Page

40, line 25, *for* $2\,a$, *read* $2\,a^2$.

61, last line *for* $\frac{a-b}{c-d}\times$, *read* $\frac{a-b}{c-d}=$

63, example 7, *for* $-\frac{3\,a\,b}{d}$, *read* $-\frac{3\,a\,b}{4\,d}$.

„ 8, *for* $b^2\,c\,d$, *read* $b^2\,c\,d\,m$.

„ 13, in the second member supply the Denominator $b\,c-2\,b\,d$.

„ 14, *for* $\frac{a+b}{27}$, *read* $\frac{135\,a^2\,b+14\,b-91\,a}{378}$.

76, „ 2, *for* $10-3x$, *read* $10+3x$.

„ 3, *for* $\frac{13-x}{3}$, *read* $\frac{12-x}{3}$.

„ 9, *for* $=\frac{x^2}{a^4+a}$, *read* $=\frac{x}{a}$.

„ 11, *for* $\frac{d}{4\,x-4\,b^4+3\,b}$, *read* $\frac{4\,b+d}{4\,a-4\,b^2+3\,b}$.

77, „ 16, in the Answer *for* $9\,a^2$, *read* $9\,a^3$.

„ 18, *read* $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$; $x=\sqrt{a^2-\left(\frac{b^3-2\,a}{3\,b}\right)^3}$.

„ 21, *for* $\left(\frac{c^2+b-a^2}{2\,c}\right)$, *read* $\left(\frac{c^2+b-a}{2\,c}\right)^2$.

„ 23, *for* $\sqrt{a^2+x}$, *read* $\sqrt{a^2+x^2}$.

„ 26, *for* $\sqrt{\frac{1}{a}}+$, *read* $=\sqrt{\frac{1}{a^2}}+$.

78, „ 33, *for* $=\sqrt{x}$, *read* $=\sqrt[3]{x}$.

136, line 9, first word, *for* member, *read* number.

Page

190, example 19, *for* $\frac{2}{n}$ the exponent to the answer, *read* $\frac{1}{n}$.

195, 68, *for* $\frac{\sqrt{3x}}{x+y} + \frac{\sqrt{x+y}}{3x}$, *read* $\sqrt{\frac{3x}{x+y}}$ $+ \sqrt{\frac{x+y}{3x}}$.

196, „ 77, 4th line, *for* $4a^3$, *read* $4a^2$.

80. last line, *for* $\frac{\sqrt{\frac{c}{2} - b \mp a\sqrt{(2b + c - a^2}}}{2}$,

read $\sqrt{\frac{\frac{c}{2} - b \mp a\sqrt{(2b + c - a^2)}}{2}}$.

199, example 92, in the numerator, *for* $\sqrt{x^2 + y^2}$,

read $\sqrt{x^2 - y^2}$.

„ next line, *for* $(x^2 + y^2)$ *read* $(x^2 + y)^2$.

271, line 12, *for* $\sqrt[5]{5^3 b^3}$, *read* $\sqrt[5]{a^3 b^3}$.

277, „ 21, *for* $\sqrt[5]{a^2 c^3}$, *read* $\sqrt[5]{a^4 c^3}$.

„ 25 and 26, *for* $\times a^{\frac{3}{3}}$ *read* $\times a^{\frac{3}{5}}$.

280, „ last line but one, *for* a^{2m}, *read* a^{3m}.

363, and the following pages, *for* figurative, *read* figurate.

414, „ last line but one, *for* ^{-r}h, *read* h^{-r}.

CHARACTERS, SIGNS, AND ABBREVIATIONS.

Algebraic.	*Greek Alphabet.*		*Weights and Measures.*
+ Positive, additive.	Α, α, Alpha.	Ν, ν, Nu.	Cwt. Hund. weight.
— Negative, subtractive.	Β, β. Beta.	Ξ, ξ, Xi.	Qr. Quarter.
× Multiplicative.	Γ, γ, Gamma.	Ο, ο, Omicron.	Oz. Ounce.
÷ Divisive.	Δ, δ, Delta.	Π, π. Pi.	Dwt. Pennyweight.
= Equal.	Ε, ε, Epsilon.	Ρ, ρ, Rho.	℔ { Pound. / *Livre.*
: :: Proportionate.	Ζ, ζ, Zeta.	Σ, σ, Sigma.	
$\succ$ Greater.	Η, η, Eta.	Τ, τ, Tau.	Ɱ *Marc.*
∠ Less.	Θ, θ, Theta.	Υ, υ, Upsilon.	℥ { Ounce. / *Once.*
∾ Difference.	Ι, ι, Iota.	Φ, φ Phi.	
∞ Infinite.	Κ, κ, Cappa.	Χ, χ, Chi.	ʒ { Dram. / *Gros, dragme.*
√ Root.	Λ, λ, Lambda.	Ψ, ψ, Psi.	
∴ Ergo.	Μ, μ, Mu.	Ω, ω, Omega.	℈ Scruple, *denier.*

LINEAR MEASURES.

1 Brit. mile =8 furlongs =320 per. =1760 yards =5280 feet =63360 inches =633600 lines.
The length of | 1 = 40 = 220 = 660 = 7920 = 79200
the seconds pendulum | 1 = $5\frac{1}{2}$ = $16\frac{1}{2}$ = 198 = 1980
at 62° Fahr. ther. in a vacuum, and | 1 = 3 = 36 = 360
at the level of the sea in London=39,13929 inches, | 1 = 12 = 120
of which 36=1 yd. & 39,37079=1 Fr. *metre.* (Comrs. Rpt. 1823,) The | 1 = 10
length of the *metre* by law established, or 0,0000001 *Quart du Merid.* | 1 milim=0,03937 *in.*
Terrestre at 16°,25 centl. ther.=0,5130740 *toi de l' Acad.* | 1 centim=10 = 0,39371 *in.*
The quarter of the meridian is properly | 1 decim = 10 = 100 = 3,93708 *in.*
5131111 *toises,* (Biot.) | 1 metre = 10 10 =1000 =39,37079 *in.*

1 decametre,	= 10	=	0 *mile*	0 *fur.*	10 *yds.*	=33,7 *inches.*
1 hecatometre,	= 100	=	0 ,,	0 ,,	109 ,,	13,
1 chiliometre,	= 1000	=	0 ,,	4 ,,	213 ,,	22,2
1 myriametre,	=10000	=	6 ,,	1 ,,	150 ,,	6,

The multiples of the French system, or, according to the French orthography, the hectometre, kilometre, &c. are named from the Greek numerals, and the submultiples are from the Latin numerals.

12 lignes = 1 pouce. | 1 *Eng. foot*=0,9383 Fr. 1 pied-de-roi=1,06575 *Eng.*
144 12 = 1 pied-de-roi. | 1 toi. de l'Acad.=6,3945 *English feet.*
844 72 6 = 1 toi. de l'Acad. | 1 pied ult. roy= $\frac{1}{3}$ metre.
3163 264 22 $3\frac{2}{3}$ = 1 per. roy (Supt. 5 Edit. Ency. Brit.)

SQUARE MEASURES.

144 inches = 1 foot | In Scotland, one square ell is equal to 9,61 square feet; and
1296 = 9 = 1 yard. | one square fall=36 sq. ells=345,96 square feet.
An Irish | $272\frac{1}{4}$ = $30\frac{1}{4}$ = 1 pole. | one square rood=40 falls; one acre
square perch is 49 yards, | 1210 = 40 = 1 rood. | 4 roods, and 1 square mile
and an Ir. acre=7850 yards. | 4840 =160 = 4 = 1 acre. | =640 acres.
∴ 1 Irish acre : 1 English acre : : 49 : 30. | 102400 = 2560 = 640 = 1 mile.

324 pieds quarrés, = 1 perche. | 1 *are*=1 *decamet.* sq. 1 *hectare*=1 *hecatometre*
32400 = 100 = 1 arpent de Paris. | =2,47117 English acres.
48400 = 100 = 1 arpent de mes. royale=1 [illegible]

45 sqr. ft.	=1 *chittack.*				1 tois quarr.=3,798744 met. qu. 1 pied quar.=0,105521 met qr.
720	= 16	= 1 *cottah.*			1 pouce quarré =0,00073278 metres quarrés.
14400	=320	=20	=1 *bigah.*		1 lieue quar.=0,1975309 myriamet. qr.
43560	=966,4	=60,4	=3,02	=1 Eng. acre.	=19,75309 myriares.

MEASURES OF CAPACITY.

2 pints.	1 quart.						The unit of capacity is the gallon containing 10 lbs. Adp·
8	= 4	= 1 gallon.					weight of distilled water weighed in air at 62°
252	= 126	= 31½	=1 barrel.				Fah. thermom., barometer at 30 inches.
504	= 252	= 63	=2	=1 hogshead.			(Report of the Commissi-
1008	= 504	126			=1 butt or pipe.		oners, 1823.)
2016	=1008	252				=1 tun.	

2 gallons	1 peck.			The gallon of 10lbs. Adp. is likewise the unit of dry measure
8	= 4	1 bushel.		for corn or other dry goods not heaped. (Report
64	=32	8	=1 quarter.	of Commissioners 1823.)

The old Paris pint is equal to 48 cub. inches, (Dict. de l'Acad.)=58,11 English inch.
The *litre* is a cubic *decimetre*, equal to 61,0279 cubic English inches, or 2⅛ wine pints nearly. A *chiliolitre*= 1 tun, 12¾ wine gallons.
1 *pied*^{3}=1,21 cub. foot nearly ; 5 *pied*^{3}=6 cub. feet nearly. 1 *stere*=1 *metre*^{3}=35,3171 cub. feet.

WEIGHTS.

1 pound Troy	= 12 ounces	=240 dwts.	=5760 grains=	Imperial standard unit of weight,
	1	= 20	= 480	of which 252,72 grains are equal in
		1	= 24	weight to one cubic inch of distilled

16 drams	= 1 ounce					water in a vacuum, at 62° of Fahren-
256	=16	=1 lb. Adp.				heit's thermometer. 7000 such grains
One cubic foot of distilled	14	= 1 stone.				= 1 lb. Avoir-du-pois. (Commission-
water weighs 62½ lbs. Avoir-	28	= 2	= 1 qr. cwt.			ers' Report, 1823.)
du-pois, or 1000 ounces	112	= 8	= 4	= 1 cwt.		
nearly.	2240	= 160	=80	= 20	=1 ton.	

1 lb. Adp.=1 lb. 11 dwts. 20 grains Troy.
1 oz. Adp. =0,91152 ounce Troy.
The Avoir-du-pois weights are used in commerce for all kinds of groceries, fruit, tobacco, butter, cheese, iron, brass, lead, tin, &c.

1 carat=4 grains=64 parts=543,226 grains Troy.

1 lb. Troy =0,82274 lb. Avoir-du-pois,
1 oz. Troy =1,09707 oz. Avoir-du-pois.
Troy weights are used for gold, silver, diamonds, costly liquors, &c. Apothecaries divide the ounce into 8 drams=24 scruples=480 grains.

FRENCH WEIGHTS.

24 grains=	1 denier or					These weights are used for gold, silver, and precious commodities.
72	= 3 scruples	= 1 gros.				The livre, or 2 marcs, is equal to 7560 *grains Troy*.
576	= 24	= 8	= 1 once.			The once is equal to 472,5 *grains Troy*.
4608	=192	= 64	= 8	=1 marc.		(Ency. Britannica.)
9216	=384	=128	=16	=2	=1 livre de Paris.	

The gramme is equal to a centimetre cube of distilled water; equal to 18 grains and $\frac{841}{1000}$ of ancient French weight$=$15,440 *grains Troy* English. One kilogramme is equal to the weight of one litre of water$=$2,1133 *lbs. Troy*.

ANCIENT WEIGHTS.

The Roman lb.$=\left\{\begin{matrix}10\\12\end{matrix}\right\}$ ounces. The Roman ounce was the Adp. ounce$=$7 denarii$=$8 drachmæ.

Egyptian.		*Eng. grs.*	*Ancient Jewish.*				*lb.*	*oz.*	*dt.*	*gr.*	*Attic.*				*lb.*	*oz.*	*dwt.*	*gr.*
mina............	=	8326			shekel	=	0	0	9	2$\frac{4}{7}$			drachma	=	0	0	2	16,9
idem Ptolemaic ..	=	8985	60		maneh	=	2	3	6	10$\frac{2}{7}$	100		mina	=	1	1	10	10,
idem Alexandrian	=	9992	3000	50	talent	=	113	10	1	10$\frac{5}{7}$	6000	60	tal.	=	67	7	5	0,

Supp. Enc. Brit. | *Crabb's Technological Dictionary.*

INDIAN WEIGHTS.

Calcutta.

Denominations.				*Factory in Avoir-du-pois.*			*Bazar in Avoir-du-pois.*		
1 sa.=						5,973 drms.			6,571 drms.
5	= 1 chittack=				1 oz.	13,867 ,,		2 oz.	0,853 ,,
80	= 16	= 1 sr. =		1 lb.	13 ,,	13,867 ,,	2 lbs.	0 ,,	13,653 ,,
3200	=640	=50	=1 md.	74 ,,	10 ,,	10,666 ,,	82 ,,	2 ,,	2,133 ,,

Thornton's E. I. Calculator.

FOR GOLD, SILVER AND PRECIOUS STONES.

1 mohur=13,28 massa=17 annas=106,25 rutties. | 1 massa =1,28 anna =8 rutt.=32 dans.

1 tolah =12,5 ,, =16 ,, =100 ,, =224,588 grs. Troy. | 1 ,, =6,25 ,, =25 ,,

1 sicca =10 ,, =12,32 ,, = 80 ,, = 179,66 grs. Troy=6,5706 drs. Adp. | 1 ,, = 4 ,,

N. B. 1 carat=1,437 rutties=1 rutty 1,75 dan. 1 rutty=0,7 carat=45 parts.

Thornton's E. I. Calculator.

Madras.					Bombay.				
40 pollums=	1 viss.			The candy is equal to	30 pice=	1 seer.			The candy is equal to
320 ,,	= 8 ,,	= 1 maund.		lbs. 500 Adp.	1200 ,,	= 40 ,,	=1 maund.		lbs. 560 Adp.
6400 ,,	=160 ,,	=20 ,,	=1 candy.		24000,,	=800 ,,	=20 ,,	=1 candy.	

Thornton's E. I. Calculator.

CHINESE WEIGHTS.

10 cash.............= 1 candarine | 16 tales =1 catty = 1 lb. 5 oz. 5,333 drms. Adp.

10 candarine ....= 1 mace | 100 catties =1 pecul =133 lb. 5 oz. 5,333 drms. Adp.

10 mace ....= 1 tale=584,64 grs. Troy=(by a different estimate) 579,84 grs.

Thornton's E. I. Calculator.

Equatorial Radius of the Earth in English feet, ... 20923713
Polar Radius,.. 20853810
Mean Radius,... 20888730
Difference of Equatorial and Polar Radius,.................................. 69903

If the Radius is considered as unity, then the semi-circumference of a circle, commonly expressed by the letter

$$\pi = 3,141592653589 \quad \&c.\ \&c.$$
$$\text{the arc of } 1^\circ = 0,017453292520$$
$$\text{the arc of } 1' = 0,000290888209$$
$$\text{the arc of } 1'' = 0,000004848137$$

Length of the Second Pendulum at London,.. ..... 39,13929 inches.
Length of the Second Pendulum at Calcutta,.. .. 39,06003 ,,

TABLE OF SPECIFIC GRAVITIES.

Solids.		Solids.	
Platina,	22,000	Sulphur,	1,810
Fine Gold,	19,400	Chalk,	1,793
Standard Gold,	17,724	Ebony,	1,177
Lead,	11,325	Pitch,	1,150
Fine Silver,	11,091	Rosin,	1,100
Standard Silver,	10,535	Mahogony,	1,063
Copper,	9,000	Amber,	1,040
Gun-metal,	8,784	Boxwood,	1,030
Fine Brass,	8,350	Butter,	0,940
Cast Brass,	8,000	Oak,	0,925
Steel,	7,850	Ice,	0,908
Wrought Iron,	7,645	Beech,	0,700
Cast Iron,	7,425	Fir or Deal,	0,550
Tin,	7,320	Cork,	0,240
Loadstone,	4,930		
Mean of the whole Earth,	4,500	**Fluids.**	
Crude Antimony,	4,000	Pure Mercury,	14,000
Diamond,	3,517	Common Do.	13,600
Granite,	3,500	Oil of Vitriol,	1,700
White lead,	3,160	Aqua fortis,	1,300
Marble,	2,705	Aqua regia,	1,234
Pebble stone,	2,700	Human blood,	1,054
Rock Crystal,	2,650	Urine,	1,032
Pearl,	2,630	Cow's milk,	1,031
Green Glass,	2,600	Sea water,	1,030
Flint,	2,570	Ale,	1,028
Common stone,	2,500	Vinegar,	1,026
Crystal,	2,210	Common water,	1,000
Clay,	2,160	Distilled water,	0,993
Oyster shells,	2,092	Red wine,	0,990
Brick,	2,000	Proof spirits,	0,931
Common earth,	1,984	Olive Oil,	0,913
Vitriol,	1,880	Pure spirits of wine,	0,866
Horn,	1,840	Oil of turpentine,	0,800
Ivory,	1,825	Common air,	0,0012

ELEMENTS

OF

ALGEBRA.

ALGEBRA has been defined to be the language of mathematical analysis. Analysis is, generally, the method of resolving mathematical problems, by reducing them into equations.

As beginners are frequently disheartened in studying a science by the difficulties presented by a long elementary theory, of which they can perceive no practical utility, we shall begin by endeavouring to shew, that Algebra is the simplest of all languages, by giving a few examples, which we shall solve at first without its aid. But if it can be shewn, that by means of the Algebraic language we arrive at the true result by a shorter and never-failing method, its utility must be evident.

Of the Method of forming an Equation from a given Problem.

A father bequeaths to his three sons the sum of Rs. 1294, on these conditions: that the eldest is to have Rs. 197 more than the second son, who is to receive Rs. 112 more than the youngest. What will be the portion of each?

If we knew one of the three portions, that of the youngest for example, we should add Rs. 112 to have the portion of the second son; to which we should have to add Rs. 197 for the share of the eldest, and the sum of all the three shares would make up the whole amount bequeathed, viz. Rs. 1294.

The three conditions may be expressed simply enough by saying—1st. The share of the youngest plus Rs. 112, is equal to that of the second son;—2nd. The share of the second son plus Rs. 197, is equal to that of the eldest;—3rd. The sum of the three shares is equal to Rs. 1294, the amount bequeathed.

These expressions are termed equations, which are composed of two equal members; the share of the youngest plus

Rs. 112, being the first member of the 1st equation; the share of the second son being the second member.

In lieu of the share of the second son in the 2nd and 3rd equations, we might substitute that of the youngest plus Rs. 112; the 2nd equation would then stand thus:—

The share of the youngest plus Rs. 112, plus Rs. 197,—or the share of the youngest plus Rs. 309,—is equal to that of the eldest son.

Lastly, the 3rd equation may be expressed as follows:

The share of the youngest son, plus the share of the youngest plus Rs. 112, plus the share of the eldest, i. e. twice the share of the youngest plus Rs. 112, plus the share of the eldest, are equal to Rs. 1294.

The preceding three equations are thus reduced to two:—If in the second we substitute for the share of the eldest that of the youngest plus Rs. 309, to which it has been shown to be equal, we obtain one single equation:

Twice the share of the youngest plus Rs. 112, plus the share of the youngest plus Rs. 309, or three times the share of the youngest son plus Rs. 421—are equal to Rs. 1294.

If from each of the two equations, we take away an equal quantity, it is evident that the two remainders will be equal; we may then write:

Three times the share of the youngest son plus Rs. 421, less Rs. 421, are equal to Rs. 1294, less Rs. 421, or three times the share of the youngest son are equal to Rs. 873, or in other words, the share of the youngest son is equal to the third part of Rs. 873.

The problem is now resolved. It is reduced to finding the third part of Rs. 873, which is Rs. 291. This number of rupees is therefore the share of the youngest son; adding to this share of the youngest son Rs. 112, we get Rs. 403, for the share of the second son; and augmenting this last by Rs. 197, we obtain Rs. 600, for the share of the eldest son; the sum of these three shares make up the whole amount bequeathed Rs. 1294.

II. The share of the eldest might have been considered as known, instead of that of the youngest, the three preceding equations would then stand thus:

1st. The share of the eldest less Rs. 197, is equal to that of the second son.

2nd. The share of the second son less Rs. 112, is equal to that of the youngest.

3rd. The sum of these three shares is equal to the whole amount Rs. 1294.

Substituting in the 2nd and 3rd equations, instead of the share of the second son, that of the eldest less Rs. 197, thus—

2nd. The share of the eldest less Rs. 197, less Rs. 112, or the share of the eldest less Rs 309, is equal to that of the youngest.

3rd. The share of the eldest, plus the share of the eldest less Rs. 197, plus the share of the eldest less Rs. 309, or three times the share of the eldest less Rs. 506, are equal to Rs. 1294. But as it is evident, that if to each of two equal quantities, an equal quantity be added, the sums remain equal, we can say: three times the share of the eldest less Rs. 506, plus Rs. 506, is equal to Rs. 1294 + Rs. 506, or three times the share of the eldest is equal to Rs. 1800, or the share of the eldest is the third part of Rs. 1800, viz. Rs 600; the same result as before.

III. The reasoning in the foregoing problem is very simple; in others, it is sometimes very complex; but all problems are generally composed of a certain number of expressions, such as: added to, subtracted from, plus, minus, equal to, &c. which are continually repeated, serving to connect the several operations; it is evident that by indicating these expressions by signs, these operations would be materially abridged, which is done as follows:

Addition is indicated by the sign +, read plus.

Subtraction is indicated by the sign —, read minus.

Multiplication is indicated by the sign ×, read multiplied by; and to indicate that a quantity is to be divided by another, the second is placed under the first, separated by a line, as $\frac{3}{4}$, signifying that 3 is to be divided by 4. Lastly, to show that two quantities are equal, the sign = is placed between the two expressions, which sign is read equal; and also the sign ∴ stands for ergo; viz. consequently.

These abbreviations, though already considerable, do not suffice; the frequent repetition of the expressions, the number to be divided, the given number, the number sought, the smaller part, &c. render the operations extremely tedious. With regard to the given quantities, the expedient which first offered itself, was to represent them, as in arithmetic, by the given numbers themselves; but as we cannot express in the same manner unknown quantities; a conventional sign, which indeed has varied with the times, has been substituted. Lastly, it was agreed to make use of the letters of the alphabet, the first, to designate known quantities, and the last letters, unknown quantities. Algebra arose from the use of these various signs.

IV. By the substitution of these signs, for their signification, in the solution of the problem, to which we have arrived by reasoning alone, it will be seen that, the one is but a translation from the common into Algebraic language. Taking the same example:

For the share of the youngest son, put x

Then the share of the second son will be, $x +$ Rs. 112

Consequently, the share of the eldest son will be, $x +$ Rs. 112 + Rs. 197

And the sum of the three shares is the whole amount bequeathed, $3x +$ Rs. 421 = Rs. 1294

And subtracting Rs. 421 from each side, we get,

$$3x = \text{Rs. } 1294 - \text{Rs. } 421 = \text{Rs. } 873$$

(thence is derived the generally given rule, that to transpose a quantity from one of the members of an equation to the other, the sign must be changed, viz: if + stands before it, it will have the sign — when transposed to the other side; and if — stands before it, it will have the sign + when transposed to the other side of the equation); or, $3x =$ Rs. 873 and dividing by 3 (for it is equally evident, that if equal quantities are multiplied, or divided by the same quantity, they remain equal):

Thence $x =$ Rs. 291;

And the share of the second son is = Rs. 291 + Rs. 112 = Rs. 403: and lastly,

The share of the eldest is = Rs. 403 + Rs. 197 = Rs. 600.

The selection of the unknown quantity is sometimes of the greatest importance, but in such simple problems as the present, it matters not which of the three shares be taken for the quantity sought.

Let the share of the second son be represented by, x

Then the share of the eldest son is expressed by $x +$ Rs. 197

And, consequently, the youngest son's share will be, $x -$ Rs. 112

And lastly, the sum of the three shares, $3x +$ Rs. 85 must be equal to Rs. 1294: by transposing Rs. 85 we have

$$3x = \text{Rs. } 1209:$$

$\therefore$ the second son's share $x =$ Rs. 403, as has been heretofore found.

V. *To divide a given Number into two parts, having a given difference.*

To arrive at the solution, let it be observed that, 1st, *the lesser part plus the difference is equal to the greater part;* consequently, if the lesser part were known, by adding to it the given difference, we should know the greater part. 2nd. *The greater part added to the lesser part, must make up the sum to be divided.*

Substituting in the last phrase for *the greater part* the equivalent expression, *the lesser part plus the difference*, it will stand thus:

The lesser part plus the difference added to the lesser part, must make up the sum to be divided.

But this phrase may be made shorter, by saying:

Twice the lesser part added to the difference, makes up the whole sum to be divided.

Thence the conclusion, that

Twice the lesser part is equal to the number to be divided, diminished by the difference; therefore:

Once the lesser part is equal to the half of the difference between the number to be divided and the given difference.

Or, what amounts to the same thing—

The lesser part is equal to the half of the number to be divided, diminished by half the given difference.

Let the given number be 40, and the difference $= 4$. Make the lesser number, x

Then, $\underline{x + 4}$ will be the greater number, and their sum $x + x + 4 =$ the given number, thence $2x + 4 = 40$, which indicates that, if 4 be added to twice the number x, it will be equal to 40, or $2x = 40 - 4$, or $2x = 36$; and, finally, $x = 18$ or half the difference, or

$$x = \frac{40}{2}$$

The number 18, however, the result of the preceding operations, agrees only with this particular example; whilst the reasoning alone, in showing that *the lesser number is equal to the half of the number to be divided, less half the difference*, teaches how the unknown number is connected with the given numbers, and furnishes a rule by which every particular case comprised within the limits of the question, can be solved.

VI. This advantage derived from reasoning alone, arises from the circumstance that no particular number is designated. The given numbers pass without alteration from one phrase to the next, whilst in considering determinate numbers they undergo changes in every step of the operation, and on arriving at the result they leave no trace behind. Nothing traces the steps how the number 18, which may be derived from an infinity of different operations, has been formed from the numbers 40 and 4.

VII. These inconveniences are avoided by representing the number to be divided, and the given difference by characters, independent of any particular value. The letters of the alphabet answer the purpose very well, and the proposed question can by their means be thus enunciated:

To divide a known number represented by s, *into two parts of which the difference is* b.

Let the lesser number as before be denoted by, x
Then the greater number will be, $x + b$
And the sum or the number to be divided, will be, $x + x + b$, or $2x + b$, and by the condition of the question $2x + b = s$; but to have the x's only on one side, b is transposed: then

$$2x = s - b,$$

and dividing both sides by 2, because x and not $2x$ is required, we get

$$x = \frac{s-b}{2} \text{ or } x = \frac{s}{2} - \frac{b}{2}.$$

The translation of this last result into the ordinary language, by the substitution of the words and phrases represented by the letters and signs which this equation contains, gives the rule already found, according to which: *to obtain the lesser of the parts sought, we must deduct from the half of the number to be divided (or* $\frac{s}{2}$ *) half the difference.*

Knowing the lesser part, the greater part is obtained by adding the given difference to the lesser. This remark is sufficient to obtain the resolution of the proposed question; but Algebra gives more, it furnishes a rule to obtain the greater part, without the help of the lesser, thus: $\frac{s}{2} - \frac{b}{2}$ being the value of the lesser, in increasing it by the difference b, the greater part will be $\frac{s}{2} - \frac{b}{2} + b$, which signifies that after having deducted from $\frac{s}{2}$ the half of b, we must add to the remainder the whole quantity represented by b or two halves of b, which amounts to increasing $\frac{s}{2}$ by the half of b or $\frac{b}{2}$. It is evident, that $\frac{s}{2} - \frac{b}{2} + b$ is the same as $\frac{s}{2} - \frac{b}{2} + \frac{b}{2} + \frac{b}{2}$ or $\frac{s}{2} + \frac{b}{2}$, and translating this expression, we learn that: *the greater part sought is equal to the half of the number to be divided, added to half the given difference.*

In the particular question (V.) the number to be divided was 40, and the excess of the one over the other part, 4; to solve it by the ordinary rules, which we have found above, we must

effect on the numbers 40 and 4, the operations which are indicated to be done on s and b.

The half of 40 is 20, and the half of 4 is 2, therefore the lesser portion is $20 - 2 = 18$ and the greater $20 + 2 = 22$.

VIII. We have designated by x the lesser of the two portions, and thence we have deduced the greater: but should this last be directly required, it may be represented by x, the other would then be $x - b$, since the lesser is equal to the greater diminished by the given difference; the number to be divided will thus be expressed by $x + (x - b)$ or $2x - b$, consequently

$$2x - b = s.$$

This result shews, that $2x$ exceeds the quantity s by the quantity b, therefore $2x = s + b$. But to get x and not $2x$, we must divide both members by 2; thence

$$x = \frac{s}{2} + \frac{b}{2}$$

the

which gives precisely the same rule as was found before for ascertaining the greater of the two parts sought. The lesser number will be obtained, by deducting the given difference (b) from the greater, viz.: $\frac{s}{2} + \frac{b}{2} - b = \frac{s}{2} - \frac{b}{2}$, as before.

The same relation between the numbers given and sought can be enunciated in several very different ways; that which has led us to the preceding question is also that which results from the following enunciation.

Knowing s *to be the sum of two numbers and* d *their difference, to find each of these two numbers*; Or, making use of other terms, the number to be divided is the sum of the two parts sought, and their difference is the excess of the greater above the lesser. This alteration in the terms of the enunciation, being applied to the rules found before (No. VII.), may be thus expressed:

The lesser of the two parts sought, is equal to half the sum less half the difference.

The greater is equal to half the sum plus half the difference.

IX. The following question is analogous to the preceding, but a little more complicated.

To divide a given Number into three parts, so that the difference of the middle part and the least shall be one given number, and the difference of the middle part and the greatest shall be another given number.

In order to fix the ideas, we shall at first assume for the known numbers, determinate values.

Be then the number to be divided 230,

and the difference between the middle and the least part 40, that between the middle and the greatest part 60.

Let x indicate the least part, then the middle can be expressed by the least part plus the difference between the two numbers, or

$$\text{middle} = x + 40,$$

and the greatest part is indicated by the middle plus the difference between the two numbers.

$$\text{Greatest part} = x + 40 + 60.$$

But the three parts together must make up the number to be divided, i. e.

$$x + x + 40 + x + 40 + 60 = 230.$$

Separating the given numbers from the unknown quantities, we get

$$3\,x = 230 - 140$$

$$\text{or } 3\,x = 90$$

$$\therefore x = \frac{90}{3} = 30.$$

Adding 40 to it we obtain the middle part

$$= x + 40 = 30 + 40 = 70,$$

and augmenting 70 by ~~10~~ we get the greatest part

$$= x + 40 + 60 = 130.$$

X. Should the known numbers be different from those given in this last example, the question might be resolved by following the same method; but then we should be obliged to repeat all the reasonings and all the operations by which we have arrived at the number 30, because nothing indicates how this number is deduced from the given numbers 230, 40, and 60. In order to render the solution independent of any particular value of the numbers and to shew how the value of the unknown quantity is formed by means of the known quantities, the problem may be enunciated, as follows:

To divide a number s *into three parts, so that the difference of the middle number and the least, be a given number* b, *and the difference of the greatest and the middle, be another given number* c.

Be the least part	x
The middle	$x + b$
The greatest	$x + b + c$

the sum of the three parts $3\,x + 2\,b + c = s$;

then by subtracting from both sides $2\,b + c$, in order that the term containing the unknown quantity x should stand alone, we have $3\,x = s - 2\,b - c$; but as the value of x alone is required, both sides of the equation must be divided by 3; therefore

$$x + 7 = 17 - x,$$
$$x + a = b - x.$$

2ndly. By addition, subtraction and multiplication, as in the equations

$$9\,x + 21 = 98 - 2\,x,$$
$$a\,x - bc = dx + cd.$$

Lastly by addition, subtraction, multiplication and division, as in the equations

$$\frac{2}{3}\,x + 3\,x - 38 = \frac{7}{12}\,x + 36$$

$$\frac{ax}{c} + bd - bx = \frac{ux}{9} + \frac{p}{s}.$$

The unknown quantity is disengaged from the additions and subtractions with the known quantities with which it is combined, by bringing to one member all the terms in which it is found ; to this end, transposing the terms of one member to the other must be known.

XIII. For example, in the equation

$$9\,x + 21 = 98 - 2\,x,$$

the term $-\,2\,x$ must be transposed from the second to the first, and $+\,21$ from the first to the second. By taking away $-\,2\,x$, we withdraw the indicated subtraction of $-\,2\,x$ in the second member ; consequently to preserve the equality, or the equation, the first member must also be augmented by $2\,x$: we shall then have,

$$9\,x + 2\,x + 21 = 98.$$

But in subtracting $+\,21$ from the first member we, diminished it by 21 ; to preserve then the equality of the two members the second member must also be diminished by 21 : the equation will then stand thus,

$$9\,x + 2\,x = 98 - 21,$$

and by effecting the indicated operations, we shall get the equation,

$$11\,x = 77.$$

Or, as we have learnt from No. 4, a term is transposed from one of the members of the equation into the other, by introducing it into the other with a contrary sign.

To reduce this rule to practice, it must first be observed that the first term of each member, when it is not preceded by a sign is always considered to have the sign + before it ; thus in transposing the first term of the second member of the literal equation, $ax-bc=dx + cd$, to the first member, we must write,

$$ax - dx - bc = dc,$$

then transposing $-\,bc$ from the first into the second member we shall have

$$ax - dx = dc + bc.$$

XIV. It is generally an easy operation to disengage the unknown quantity when it is involved with known quantities by addition or subtraction only, this is done by bringing all the terms that contain the unknown quantity to one member of the equation, should the unknown quantity, as in the present example, have a factor or multiplier, it may always be resolved into two factors, of which the one shall contain only given quantities, and the other the unknown quantity.

This simplification presents itself whenever the proposed equation is numerical and contains no fraction, for in this case all the terms that contain the unknown quantity are reduced to one. For example, the equation $12x + 8x - 3x = 5x + 30 - 6$. In effecting the operations indicated, we shall find successively,

$$17x = 5x + 24,$$
$$12x = 24;$$

but $12x$ can be resolved into two factors 12 and x, the unknown factor x, is obtained by dividing it by the factor 12, and dividing also the other member by 12, the equality of the two members remains, since by multiplying or dividing equal quantities by the same or an equal quantity, they remain equal, hence:

$$x = \frac{24}{12} \text{ or } 2$$

This resolution is effected in the same way in literal equations as:

$$ax = bc;$$

but since the terms ax signifies the product of a by x, we must divide both sides by a, we then have:

$$x = \frac{bc}{a}$$

In the preceding equation

$$ax - dx = dc + bc$$

where x is a factor common to a and to b, it may be written thus:

$$x(a - d) = c(b + d),$$

as in the numerical example just given; the first member having two factors, the unknown factor x can be divided by the known factor $a - d$; and dividing also the second member by the same quantity $a - d$, in order to maintain the equality, we find

$$x = \frac{c(b + d)}{a - d}$$

Let there be the equation,

Let there be the equation

$$a x - b x + b c x = a b - a c$$

which has three terms containing the unknown quantity x; as $a x$, $b x$ and $b c x$ represent the respective products of x by the quantities a, b, and the product $b c$, the expression $a x - b x + b c x$ translated into ordinary language, gives the following phrase:

From x *taken as many times as* a *contains units, subtract as many times* x *as there are units in* b, *then add to the result the same quantity* x, *as often as the product of* b *by* c *contains units, the whole will be equal to the product of* a *by* b *diminished by that of* a *by* c.

Whence it appears that the unknown quantity x is taken as often as there are units, in the difference of the numbers a and b together with the product of b c, viz: as often as is indicated by $a - b + b c$, the two factors of the first member are consequently $a - b + b c$ and x; whence we have

$$x = \frac{a b - a c}{a - b + b c} \text{ or } \frac{a (b - c)}{a - b + b c}.$$

This reasoning, which may be applied to any other example, shews that *after collecting into one member only, the different terms containing the unknown quantity; the factor which multiplies this unknown quantity is formed of all those quantities which multiply it separately with the same signs with which they are preceded; and the unknown quantity is obtained, by dividing the member composed of known quantities alone, by that same factor.*

Agreeably to the preceding rule, the equation $a x - 7 x = a b$ gives:

$$x = \frac{a b}{a - 7}.$$

For the same reason, the equation $x + b x = a + c$, gives

$$x = \frac{a + c}{1 + b}$$

for it ought to be observed, that a quantity may always be regarded as multiplied or divided by unity. It is obvious that in $x + b x$, the unknown quantity is contained once more than in $b x$, it is therefore multiplied by $b + 1$.

XV. Should all the terms of an equation have one common factor, this common factor may be suppressed without affecting the equality, since it will be the same thing as dividing by the same number all the terms of the two quantities which are supposed equal to each other.

For example, let the equation be,

$$3\,a\,b\,x - 12\,b\,c = 9\,b\,d\,x + 6\,a\,b\,d,$$

we observe that the numbers 3, 12, 9, and 6, are all divisible by 3, suppressing then this factor, the reduction of this equation will stand thus:

$$a\,b\,c\,x - 4\,b\,c = 3\,b\,d\,x + 2\,a\,b\,d.$$

Observing further, that the letter b is combined in all its terms by multiplication, indicates, that b is a common factor to all the terms: it may therefore also be suppressed and we shall then obtain:

$$a\,c\,x - 4\,c = 3\,d\,x + 2\,a\,d.$$

In applying to this equation the rule found in No. 12 and No. 13 we shall have successively,

$$a\,c\,x - 3\,d\,x = 2\,a\,d + 4\,c$$

$$x = \frac{2\,a\,d + 4\,c}{a\,c - 3\,d}.$$

XVI. We may now proceed to the equation of which the terms have divisors. The preceding rules might be immediately applied to it, whenever the unknown quantity is not contained in the denominators; but in this case even it is often more simple to bring all the terms to the same denominator which may then be expunged.

Let there be the equation,

$$\frac{2}{3}\,x + 8 - \frac{3}{4} = \frac{7}{8}\,x + 16 - \frac{5}{6}\,x.$$

Note. As the Rule generally given for adding and subtracting fractions is by some persons considered operose, it may perhaps not be amiss to introduce another mode of arriving at the same end, which some learners might adopt as both shorter and easier. A number which is contained in another any number of times without leaving a remainder is (as is well known) called an *aliquot* part of that number, or simply aliquot of that number. Thus 7 being contained 3 times in 21, without leaving a remainder, is called an *aliquot* part of 21, but the same number 7 is called an *aliquant* part of 23, because there is no number which being multiplied by 7 will produce 23. A number which is aliquot of another is also aliquot of any of its multiples; as 4 aliquot of 12 is also aliquot of 24, 36, 48, &c. Again a number may not be exactly contained in one or more numbers separately, yet it may be aliquot of their product: as for example, 9 is not contained in 3 nor is it aliquot of 12, but it is aliquot of their product 3×12; nor is the same number 9 exactly contained in 2, 3 or 12 yet 9 is aliquot of their product 72. With a little attention it may easily be recognized if a number is aliquot of the product of two or more numbers, though it may not be so separately of each, as all of these numbers have a common divisor.

Let it be required to add $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{5}{8}$; first, draw lines or, imagine such to be drawn, through the denominators of these fractions which

Arithmetic gives a rule for the reduction of fractions to the same denominator, and also to convert whole numbers into equivalent fractions of any denominator. we shall transform by that rule into fractions, having the same denominator, all the terms of the proposed equation, viz. to multiply the numerator of each term, by all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator.

$$\frac{2\times4\times8\times6}{3\times4\times8\times6}x+\frac{8\times3\times4\times8\times6}{3\times4\times8\times6}-\frac{3\times3\times8\times6}{4\times3\times8\times6}=$$
$$\frac{7\times3\times4\times6}{8\times3\times4\times6}x+\frac{16\times3\times4\times8\times6}{3\times4\times8\times6}-\frac{5\times3\times4\times8}{6\times3\times4\times8}x$$

Each fraction having now the same denominator, it may be cancelled, since by doing so, we multiply every part of the equation by this denominator, which therefore does not destroy the equality of the members. It will then become,

$2\times4\times8\times6\times x+8\times3\times4\times8\times6-3\times3\times8\times6=$
$7\times3\times4\times6\times x+16\times3\times4\times8\times6-5\times3\times4\times8\times x$ or,

$$384\,x\times4608-432=504\,x\times9216-480\,x$$

---

are aliquot of the denominators of others, or of the product of two or more; as 2 and 3 are aliquot of 4 and 6; and 8 of the product of 4 by 6, thus:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{3}{5}+\frac{5}{6}+\frac{5}{8}$$

the least common denominator is the product of the remaining denominators, $4\times5\times6=120$, disposing then of these fractions in a vertical line, and writing the common denominator over it, thus:

120

$\frac{1}{2}$	the denominator 2,	is cont. in the com. denominator 60	times, wh. mult. by its numerat. $1=60$
$\frac{2}{3}$	. . 3,	 40	. . . . . $2=80$
$\frac{3}{4}$	. . . 4,	 30	. . . . . $3=90$
$\frac{3}{5}$	. . . 5,	 24	. . . . . $3=72$
$\frac{5}{6}$	. . . 6,	 20	. . . . . $5=100$
$\frac{5}{8}$	. . . 8,	 15	. . . . . $5=75$

By addition we get, ... 477

having the common denominator 120, viz.

$$\frac{477}{120}=3\,\frac{117}{120}$$

reducing, $360\ x = 5040$

$$x = 14$$

We may obtain the result by a shorter process.

Let it be required to find the value of x in the equation.

$$\frac{3\,x}{4} + \frac{2\,x}{3} + \frac{5\,x}{6} + \frac{7\,x}{8} = 150\,;$$

the denominator 3 being aliquot of 6, and that of 8 of the product of 4 by 6, the least common denominator is the product of the remaining two denominators $4 \times 6 = 24$, the equation can therefore be transformed into the following by multiplying every term by 24:

$$\frac{72}{4}\,x + \frac{48}{3}\,x + \frac{120}{6} + \frac{168}{8} = (18+16+20+21)\,x = 150 \times 24 \text{ or}$$

$$75\,x = 3600, \text{ and finally}$$

$$x = 48.$$

Find the value of x in the equation

$$70\tfrac{1}{2}\,x + 26 = 92\tfrac{1}{3}\,x + 2\tfrac{5}{6}\,x - 122, \text{ or}$$

by multiplying every term by 6

$$423\,x + 156 = 554\,x + 17\,x - 732. \text{ Transposing}$$

$$732 + 156 = (554 + 17 - 423)\,x,$$

and by effecting the operations indicated, $888 = 148\,x$, lastly dividing by 148

$$x = 6.$$

Be it required to add

$\frac{2}{3} + \frac{1}{4} + \frac{5}{6} + \frac{3}{7} + \frac{7}{12} + \frac{3}{14}$ by the rule explained in this note

the common and least denominator, is 84

$\frac{2}{3}$	...	28	...	56
$\frac{1}{4}$	...	21	...	21
$\frac{5}{6}$	...	14	...	70
$\frac{3}{7}$	...	12	...	36
$\frac{7}{12}$	...	7	...	49
$\frac{3}{14}$	...	6	...	18

$$\frac{250}{84} = 2\,\frac{82}{84} = 2\,\frac{41}{42}.$$

Should it be required to add fractions having different signs as + and —, or what is the same thing, to subtract fractions from other fractions, we must proceed as is done in the following example :

The same process is applicable to literal equations, it should however be observed, that the multiplications, divisions, &c. which may always be performed in numbers, can sometimes only be indicated by letters. For example, let there be the equation

$$\frac{a\,x}{b} - \frac{d}{e} = \frac{c\,x}{e} + \frac{a\,f\,d}{eh}$$

the common and least denominator is visibly $b \times e \times h$; wherefore, multiplying every term by it, the equation will thus be transformed into the following:

$$a\,e\,h\,x - b\,h\,d = b\,c\,h\,x + a\,b\,f\,d, \qquad \text{or}$$
$$x\,(a\,e\,h - b\,c\,h) = a\,b\,f\,d + b\,h\,d, \text{ and finally}$$
$$x = \frac{b\,d\,(a\,f + h)}{h\,(a\,e - b\,c)}$$

$$\frac{1}{2} + \frac{3}{4} - \frac{5}{7} + \frac{3}{5} - \frac{2}{3} - \frac{4}{15} + \frac{5}{6} - \frac{3}{10} - \frac{5}{12}$$

the denominators 2, 5 and 3 being aliquot of 4 and 15; 6, 10, and 12, of the product of 4 by 15, the least common denominator is consequently the product of the remaining denominators 4, 7 and 15 = 420, disposing of the fractions in the same manner, as in the preceding example, observing only to add first all the + fractions and deducting from it the sum of the — fractions, viz :

		420			
$\frac{1}{2}$	...	210	...		210
$\frac{3}{4}$	...	105	...		315
$\frac{3}{5}$	...	84	...		252
$\frac{5}{6}$	...	70	...		350
					1,127
$\frac{5}{12}$	...	35	...	175	
$\frac{5}{7}$	...	60	...	300	
$\frac{3}{10}$	...	42	...	126	993
$\frac{2}{3}$	...	140	...	280	
$\frac{4}{15}$	...	28	...	112	
					134
				Remains,	420

Of the Addition of Algebraic Quantities.

XVII. Quantities which contain but one term are called *monomials*, as:

$$+ 3\,b, - 4\,a\,b, \text{\&c.}$$

Binomials are those that have two terms, as :

$$a + b;\ 2\,a - 3\,b;\ 7\,b - 3\,x, \text{\&c.}$$

Trinomials are those which have three terms, those that have more than three terms are generally called *polynomials*.

The addition of monomials or commonly called *simple* quantities, is performed by writing them one after the other with the sign + between them; thus, $a + b$ is to express that b is to be added to a. But when it is proposed to add together several algebraic expressions, we endeavour at the same time to simplify the result, by reducing it to the least possible number of terms by uniting several in one.

This union of terms is done in Nos. 7 and 9, by reducing the quantity $x + x$ to $2\,x$, and the quantity $x + x + x$ to $3\,x$. It can take place only with respect to quantities which are expressed by the same letters and which are for this reason called *similar* quantities. A literal quantity repeated a certain number of times, is regarded as a unit; thus, $2\,a$ and $3\,a$ are quantities considered as 2 or 3 units of a particular kind, form when added, $5\,a$, or 5 units of the same kind. For the same reason, $4\,a\,b$ and $5\,a\,b$ make $9\,a\,b$.

In this case, the addition is performed on the numerical figures which precede the literal quantity indicating how often it is to be repeated. These ciphers are called *co-efficient*. The co-efficient then is the multiplier of the quantity before which it is placed ; and it should be recollected, that where no co-efficient is expressed, unity is understood, for $1 \times a$ is the same thing as a.

XVIII. If it is proposed to add any quantities, as:

$$4\,a + 5\,b \text{ and } 2\,c + 3\,d,$$

the sum total is evidently to be composed of all the parts joined together ; we must write then

$$4\,a + 5\,b + 2\,c + 3\,d.$$

If on the other hand, we have to add

$$4\,a + 5\,b \text{ and } 2\,c - 3\,d,$$

the sign -- must be retained in the sum, to indicate as subtractive, the quantity $3\,d$, which, as it is to be taken from $2\,c$, must necessarily diminish by so much the sum formed by uniting $2\,c$ with the first of the quantity proposed ; we shall then have

$$4\,a + 5\,b + 2\,c - 3\,d.$$

It is evident from these two examples, that: *in algebra the addition of polynomials is performed, by writing in order one after the other, the several quantities which are to be added, with their respective signs, observing that the terms which have no sign prefixed, are considered to have the sign + before them.*

The above operation is in fact an indication only, by which the union of two compound quantities is reduced to an addition and subtraction of a certain number of simple quantities; but if the quantities to be added contained similar terms, these terms might be united by performing the operation immediately on their co-efficient.

Let there be for example, the quantities

$$4a + 9b - 2c$$
$$2a - 3c + 4d$$
$$7b + c - e$$

the sum indicated would be, according to the rule just given,

$$4a + 9c - 2c + 2a - 3c + 4d + 7b + c - e$$

but the terms $4a + 2a$, being formed of similar quantities, can be united into one, equal to $6a$. Again the terms $9b$ and 7 give $+16b$.

The terms $-2c$ and $-3c$ being both subtractive, produce on the whole the same effect, as the subtraction of a quantity equal to their sum or of $5c$; but as there remains the term $+c$ to be added, the result will be $-5c + c$, or that there remain but $4c$ to be subtracted.

The sum of the proposed expressions, then, will stand thus:

$$6a + 16b - 4c + 4d - e.$$

XIX. This last operation, by which all similar terms are united into one, whatever sign they have is called *reduction. It is performed by summing up all similar quantities having the sign + and taking also the sum of similar quantities having the — sign; then subtracting the less of these two sums from the greater, and giving to the remainder the sign of the greater.*

It should be remarked that reduction is applicable to all algebraic operations.

To exercise the students, we give some examples of addition with their answers.

1st. To add the following quantities:

$$7m + 3n - 14p + 17r + 15q$$
$$3a + 9n - 11m + 4q + 2r$$
$$5p - 4m + 8n$$
$$11n - 2b - m - r + s - 12q$$

Answer, $7m + 3n - 14p + 71r + 15q + 3a + 9n - 11m + 4q + 2r + 5p - 4m + 8n + 11n - 2b - m - r + s - 12q$.

Making the reduction, this quantity is changed into the following:

$-9m + 31n - 9p + 18r + 7q + 3a - 2b + s$, or to begin with a term having the sign $+$

$$31n - 9m - 9p + 18r + 7q + 3a - 2b + s.$$

2nd. To add the quantities:

$$\begin{array}{l} 11bc + 4ad - 8ae + 5cd + 10ef \\ 8ae + 7bc - 2ad + 4mn - 12ef \\ 2cd - 3ab + 5ac + 4an - 3ef \\ 9an - 2bc - 2ad + 5cd + 5ef \\ \hline \end{array}$$

Answer, $11bc + 7bc - 2bc + 4ad - 2ad - 2ad - 8ac + 8ac + 5ac + 5cd + 2cd + 5cd + 10ef - 12ef - 3ef + 5ef + 4mn - 3ab + 4an + 9an$.

By reduction, this quantity becomes:

$$16bc + 5ac + 12cd + 4mn - 3ab + 13an.$$

Of the Subtraction of Algebraic Quantities.

XX. The subtraction of a monomial or simple quantity is indicated by prefixing the conventional sign $-$ before the quantity to be subtracted.

To subtract b from a is indicated by $a - b$.

When the quantities are similar, the subtraction is performed directly by means of the co-efficients.

Thus if from $12a$, we are to subtract $10a$, we have for remainder $2a$.

With regard to the subtraction of polynomials, we must distinguish two cases.

1st. If the terms of the quantity to be subtracted have the sign $+$ prefixed before each of the terms, it is obvious that this is done by giving to each the sign $-$, since it is required to deduct successively all the parts of the quantity to be subtracted.

For example, if from $5a + 12b - 3c$ were to be taken

$$3d + 5e + 7f$$

we must write $5a + 12b - 3c - 3d - 5e - 7f$.

2nd. Should the quantity to be subtracted contain terms having the sign $-$, we must give them the sign $+$. Indeed, if from the quantity a we would take $b - c$, and should first write $a-b$, we should thus diminish a by the whole quantity b; but the subtraction ought to have been performed after having first diminished b by the quantity c; we have taken therefore this last quantity too much by a quantity c which must therefore

be restored to it with the sign $+$ which gives for the true result

$$a - b + c.$$

This reasoning, which can be applied to all similar cases shews, that the sign $-$ of c must be changed into the sign $+$; and connecting this result with the preceding, we conclude that *the subtraction of algebraic quantities is performed, by writing each term of the quantity to be subtracted, with the sign*

To some readers the following illustration of the generally given rule, to change all the signs of the terms of the quantity to be subtracted, may perhaps be more satisfactory; it is regarding a mercantile account current as a collection of all the plus terms on one side under the head of creditor, and all the minus terms on the other side under that of debtor and adding the difference of the sums to that side which gives the lesser amount under the name of balance; for example.

Cr.	A.	Dr.
$+\ 50$	...	$-\ 105$
$+\ 65$	...	$-\ 50$
$+\ 105$	Balance in A.'s favor,	$-\ 65$
220		220

Every item on the Cr. side can be represented, as in this example, by the sign $+$ and those on the Dr. by that of $-$. Supposing now the $-\ 50$ is taken from the Dr. side, the Balance in A.'s favor, would then be increased by that quantity, viz. it would be $65 + 50 = 115$; the account would then stand thus:

Cr.	A.	Dr.
$+\ 50$	...	$-\ 105$
$+\ 65$	Balance in A.'s favor,	$-\ 115$
$+\ 105$	...	...
220		220

The same may be shewn by giving a literal example, viz.

Cr.	A.	Dr.
$+\ a$	...	$-\ b$
$+\ c$	...	$-\ a$
$+\ b$	Balance in A.'s favor,	$-\ c$
$a + b + c$		$-\ (a + b + c)$

and subtracting the minus quantity a from the account, it would stand thus:

Cr.	A.	Dr.
$+\ a$	..	$-\ b$
$+\ c$	Balance in A.'s favor,	$-\ (a + c)$
$+\ b$		...
$a + b + c$		$-(a + b + c)$

here again, the quantity $-\ a$, which is subtracted from the account, increases the balance; the same quantity a must therefore be added to it, that is, it must be expressed by $+\ a$. These examples suffice to shew, why a minus quantity becomes plus, when it is to be subtracted.

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changed from + into — and from — into + immediately after the quantity which is thus to be diminished.

Having written the result agreeably to this last rule, the quantities are to be reduced, when they will admit of it, according to the precept given in art. 18, this may be seen by a few examples.

1st. To subtract from $17\,a + 2\,m - 9\,b - 4\,c + 23\,d$
the quantity $51\,a - 27\,b + 11\,c - 4\,d$

Result,
$17\,a + 2\,m - 9\,b - 4\,c + 23\,d - 51\,a + 27\,b - 11\,c + 4\,d.$

By reduction this quantity will become
$$2\,m - 34\,a + 18\,b - 15\,c + 27\,d.$$

2nd. To subtract from $5\,a\,c - 8\,a\,b + 9\,b\,c - 4\,a\,m$
the quantity $8\,a\,m - 2\,a\,b + 11\,a\,c - 7\,c\,d$

Result,
$5\,a\,c - 11\,a\,c - 8\,a\,b + 2\,a\,b + 9\,b\,c - 4\,a\,m - 8\,a\,m + 7\,c\,d$ when reduced it becomes $-6\,a\,c - 6\,a\,b + 9\,b\,c - 12\,a\,m + 7\,c\,d$
or writing one of the + terms first
$$9\,b\,c - 6\,a\,c - 6\,a\,b - 12\,a\,m + 7\,c\,d.$$

Of the multiplication of Algebraic quantities.

XXI. So far as letters are only considered as expressing the numerical values of the quantities for which they stand, multiplication in algebra is to be regarded as multiplication in arithmetic. *Thus, to multiply* a *by* b *is to compound with the quantity represented by* a, *another quantity* b *in the same manner as the quantity* b *is compounded with unity.*

In No. 2, it has already been shewn what signs have been agreed upon to indicate multiplication; and that the product of a by b is expressed by $a \times b$, or by $a.b$, or only ab.

It is most frequently required to indicate several successive multiplications, as that of a by b, then that of the product $a\,b$ by c, again this last product by d and so on. In this case, it is evident, that the last result is a number having the several numbers a, b, c, d, as *factors*; and to give a general expression of this method, *the product is indicated by writing the several factors composing it in order one after the other without any signs between them*; in this manner, we have the expression,

$$a\,b\,c\,d.$$

(It is customary to arrange the letters, according to their rank in the alphabet.)

Reciprocally, every expression, such as $a\ b\ c\ d$, formed of several letters written in order one after the other, indicates always the product of the number represented by these letters.

We have already made use of those methods in which the numerical coefficients are included, since they are evidently factors of the proposed quantity. Indeed, $15\ a\ b\ c\ d$, signifying the quantity $a\ b\ c\ d$, taken 15 times, expresses also the product of the five factors

$$15,\ a,\ b,\ c,\ d.$$

XXII. It follows from this that, to indicate the multiplication of several simple quantities, such as $4\ a\ b\ c$, $5\ d\ e\ f$, $3\ m\ n$, it is necessary to write these quantities in order one after the other without the interposition of any signs, as :

$$4\ a\ b\ c\ 5\ d\ e\ f\ 3\ m\ n,$$

but as it is indifferent in which order the factors of a product are written, as its value is not altered by any change of place of the factors, we may avail ourselves of this principle, and place the numerical factors immediately after each other, as :

$$4.\ 5.\ 3\ a\ b\ c\ d\ e\ f\ m\ n,$$

and effecting the multiplication indicated by the common arithmetical rule, we get simply

$$60,\ a\ b\ c\ d\ e\ f\ m\ n.$$

XXIII. The expression of a product is considerably abridged when it contains equal factors. Instead of writing several times in succession, the letter which represents one of the factors, it is only written *once indicating by a number how many times it ought to have been written as a factor*; but as this number indicates successive multiplications, it must be carefully distinguished from a coefficient which only indicates additions ; for this reason, it is placed in smaller figures on the right of the letter and a little above it, whilst the coefficient is always placed to the left of the letter and on the same line.

Agreeably to this method, the product of a by a which according to No. 20 would be indicated by $a\ a$, becomes a^2. This 2 raised, indicates that the letter a is twice a factor in the proposed expression ; it must not be confounded with $2\ a$, which is only an abreviation of $a + a$. To render the error more evident which would arise from mistaking the one for the other, it suffices to substitute numbers instead of the letters. For example, if we have $a = 6$, $2\ a$ would become $2 \times 6 = 12$, and $a^2 = a \times a = 6 \times 6 = 36$.

Extending this method we should denote a product in which a is three times a factor by writing a^3 instead of $a\ a\ a$, for the same reason a^5 equivalent to $a\ a\ a\ a\ a$, represents a product in which a is five times a factor.

XXIV. The products thus formed by the successive multiplications of the same quantity are generally called *powers* of that quantity.

The quantity itself as a, is called the 1st power of a.

The quantity multiplied by itself as $a\ a$ or a^2; is the second power of a, or also called the *square* of a*.

The quantity multiplied twice by itself $a\ a\ a$ or a^3 is the 3rd power of a and is called also the *cube* of a.

In general, any power whatever is designated by the number of equal factors from which it is formed: a^6 or $a\ a\ a\ a\ a\ a$ is the 6th power of a.

To show the application of these denominations, we shall take the number 4, and we have

1st Power,	4		
2nd ditto,	$4 \times 4 =$	16	
3rd ditto,	$4 \times 4 \times 4 =$	$16 \times 4 =$	64
4th ditto,	$4 \times 4 \times 4 \times 4 =$	$64 \times 4 =$	256
5th ditto,	$4 \times 4 \times 4 \times 4 \times 4 =$	$256 \times 4 =$	1024
	&c. &c. &c.		

The number which indicates the power of another is called the *exponent* of that number.

When the exponent is equal to unity, it is not written; thus a is the same thing a^1.

By what precedes it is clear that *to form the power of any number it is necessary to multiply this number by itself as many times less one as there are units in the exponent of the power.*

XXV. As the exponent denotes the number of equal factors which form the expression of which it is a part, and as the product of two quantities is composed of all the factors which make up each of the two quantities, it follows that the expression a^5, in which a is 5 times a factor, multiplied by a^4, in which a is 4 times a factor, must give a product in which a is 9 times a factor, expressed therefore by a^9, and that in general, *the product of two powers of the same number has for its exponent the sum of those of the multiplicant and the multiplier.* For applying the observation in No. 21, to multiply $a\ a\ a\ a$ by $a\ a\ a\ a\ a$, is to write these quantities immediately after one another as: $a\ a\ a\ a\ a\ a\ a\ a\ a$ or a^9.

XXVI. It follows from this *that if two or more simple quantities have common letters, the expression of the product of those quantities can be shortened by adding together the*

* The denominations *square* and *cube* proceeding from geometrical considerations breaking the link of uniformity in the nomenclature of the products formed by equal factors, are very improperly used in Algebra, but on account of their shortness are frequently employed.

exponents of the similar letters ; for example, the expression of the product of the quantities $a^2\ b^3\ c^4\ d$ and $a^3\ b^4\ c^2\ d$, which would be $a^2\ b^3\ c^4\ d\ a^3\ b^4\ c^2\ d$, is abridged by collecting together the factors designated by the same letter which gives $a^5\ b^7\ c^6\ d^2$ by writing a^5 instead of $a^2\ a^3$

b^7 .... $b^3\ b^4$

c^6 .... $c^4\ c^2$

d^2 .... $d^1\ d^1$.

XXVII. As the powers are distinguished by the number of equal factors from which they are formed, so also any product whatever is denoted by the number of single factors or *primes* which produce them, we shall give to these expressions the name of *degrees*. The produce of $a^3\ b^2\ c^4$ is of the 9th degree, because it contains 9 simple factors, viz : 3 factors a, 2 factors b, and 4 factors c. It is evident that the factors a, b and c, regarded here as prime, are only so with respect to Algebra, which does not permit us to decompose them ; they may, however, represent compound numbers as well as prime ; their general value only is here considered*. Coefficients, expressed in numbers, are not considered in estimating the degree of algebraical quantities ; we have regard only to the letters.

It is evident (Nos. 21 and 25) that when two or more simple quantities are multiplied the one by the other, the number which marks the degree of the product, is the sum of those which mark the degree of each of the simple quantities.

XXVIII. The multiplication of compound quantities is reduced to that of simple quantities in considering each term of the multiplicand and multiplier by itself ; as in arithmetic, we perform the operation upon each figure of the numbers which it is proposed to multiply. The union of the partial products make up the total product. But Algebra presents a circumstance which is not found in numbers. These contain no terms to be taken away, or subtracted ; the units, tens, hundreds, &c. of which they consist are always considered as added together, and it is very evident that the total product must be composed of the sum of the products of each part of the multiplicant by

* In conformity to the analogy pointed out in the note to page 28 the term *dimensions*, is generally applied to what we here call *degrees ;* the expression above cited, will in ordinary language, have 9 *dimensions*. This example sufficiently proves the absurdity of the ancient nomenclature, borrowed from the circumstance, that the products of 2 or 3 factors, measure respectively the areas of the surfaces and of the volumes of bodies, the former of which have two, and the latter three dimensions ; but beyond this limit, the correspondence between the algebraic expressions and geometrical figures ceases, as extension can only have three dimensions.

each part of the multiplier. The same is true of literal expressions when all the terms are connected together by the sign $+$.

$$\begin{array}{ll} \text{The product of} & a + b \\ \text{multiplied by} & c \\ \hline \text{is} & a\,c + b\,c \end{array}$$

and is obtained by multiplying each part of the multiplicand by each part of the multiplier, and then adding together the two partial products $a\,c$ and $b\,c$. The operation is the same when the multiplicand contains more than two parts.

If the multiplier is the sum of several terms, it is obvious that the product is then made up of the sum of the products of the multiplicand by each term of the multiplier.

$$\begin{array}{ll} \text{The product of} & a + b \\ \text{multiplied by} & c + d + e \\ \hline \text{is} & \left\{\begin{array}{l} a\,c + b\,c \\ a\,d + b\,d \\ a\,e + b\,e \end{array}\right. \end{array}$$

for multiplying first $a + b$ by c, the 1st term of the multiplier, we obtain $a\,c + b\,c$; then multiplying $a + b$ by the second term d, we get $a\,d + b\,d$; and lastly by multiplying $a + b$ by e, the 3rd term, we find $a\,e + b\,e$; and the sum of these three results is

$ac + bc + ad + bd + ae + be$, or $a\,(b + c + d) + b\,(c + d + e)$ for the total product.

XXIX. When the multiplicand contains parts to be subtracted, the product of these parts by the multiplier must be taken from the other products, viz. they must have the sign $-$ prefixed before them. For example,

$$\begin{array}{ll} \text{the product of} & a - b \\ \text{multiplied by} & c \\ \hline \text{is} & a\,c - b\,c \end{array}$$

for each time that we take the entire quantity a, which was to have been diminished by b before the multiplication, we take the quantity b too much; the product ac, therefore, in which the whole of a is taken as many times as is indicated by the number c, exceeds the product sought by the quantity b taken as many times as is indicated by the number c, that is by the product bc; we must therefore subtract bc from ac, which gives, as above

$$a\,c - b\,c.$$

The same reasoning will apply to each of the parts of the multiplicand that are to be subtracted, whatever may be the

number and whatever may be that of the terms of the multiplier, provided they have all the sign $+$, recollecting that the terms that have no signs are considered as positive or having the sign $+$; we see by these examples, that the terms of the multiplicand affected by the sign $+$ give a partial product affected by the sign $+$, while those which are affected by the sign $-$ give a product having the sign $-$. It follows from this, *that when the multiplier has the sign $+$ the product has the same sign as the corresponding part of the multiplicand.*

XXX. The contrary takes place when the multiplier contains subtractive or negative parts; the products arising from these parts must be prefixed with a sign contrary to that they would have had by the above rule. An example will illustrate this.

Be the multiplicand	$a - b$
and the multiplier	$c - d$

the product will be $\left\{\begin{array}{l} a\,c - b\,c \\ -a\,d + b\,d \end{array}\right.$

for the product of the multiplicand $a - b$, by the 1st term of the multiplier c, will be by the last example, $a\,c - b\,c$, but by taking the whole of c for the multiplier, instead of c diminished by the quantity d, we take the quantity $a - b$ so many times too much as is denoted by the number d; so that the product $a\,c - b\,c$ exceeds that sought by the product $a - b$ by d, and this last, by what has been said is $a\,d - b\,d$, and in order to subtract it from the first, we must change the signs. (N. 20.) We have then for the result required

$$a\,c - b\,c - a\,d + b\,d.$$

XXXI. As generally only rules without any investigations are given to show that minus multiplied by minus produces plus, we shall endeavour to shew the same thing in a different way. There was no difficulty in the multiplication of $a + b$ by c, the product could only be $a\,c + b\,c$; but it was perhaps not so well understood how the product of $a - b$ multiplied by c is $a\,c - b\,c$. Suppose $a - b = d$, or $a = b + d$

multiplying each number by c

the product will be $a\,c = b\,c + d\,c$

and subtracting from each side $b\,c$, we get

$$a\,c - b\,c = d\,c$$

whence it appears, that $a\,c - b\,c$ is the product of d by c, or of $a - b$ by c, or that a quantity having the sign $+$, multiplied by a quantity with the $-$ sign, as $+c$ by $-b$, or a quantity with the $-$ sign, multiplied by a quantity having the sign $+$,

as $-b$ by $+c$, must necessarily have the sign $-$ before its product.

The same equation $ac = bc + dc$ may be written thus:

$$-dc = -ac + bc$$
$$\text{or } -dc = -c\,(a - b)$$

From this last equation it appears, that $a - b$ multiplied by $-c$, gives the product $-ac + bc$; we can thence conclude that $-b$ multiplied by $-c$ gives a product with the $+$ sign, and in general that from the observations in this number when both the factors have either the sign $+$ or $-$ before them, the product will be positive; but if one of the two factors, has the sign $+$ and the other the sign $-$ the product will be negative; or which may be expressed in different words, saying: *the product shall have the sign + when both factors have the same sign, but the sign — when the two factors have unequal signs.*

XXXII. The following example offers the applications of all these rules.

$$\begin{array}{rl} \text{Multiply} & 5a^4 - 2a^3b + 4a^2b^2 \\ \text{by} & a^3 - 4a^2b + 2b^3 \end{array}$$

$$\text{Partial products,}\begin{cases} 5a^7 - 2a^6b + 4a^5b^2 \\ -20a^6b + 8a^5b^2 - 16a^4b^3 \\ +10a^4b^3 - 4a^3b^4 + 8a^2b^5 \end{cases}$$

Reduced result, $5a^7 - 22a^6b + 12a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5$

the 1st line of partial products contains those of all the terms of the multiplicand multiplied by the first term a^3 of the multiplier; this term being considered as having the sign $+$ before it, the products which it gives, have all the same signs as the corresponding terms of the multiplicand. (N. 31.)

The first term of the multiplicand $5a^4$ being considered as having the sign $+$, the sign of the first product is omitted, which would necessarily be $+$; the coefficient 5 of a^4 being multiplied by the coefficient 1 of a^3 gives 5 for the coefficient of its partial product; the sum of the two exponents of the letter a is $4 + 3$ or 7, therefore the first partial product is $5a^7$. (N. 25.)

The 2nd term $-2a^3b$ of the multiplicand having the sign $-$, the product has also the sign $-$; the coefficient 2 of a^3b, multiplied by the coefficient 1 of a^3, gives 2 for the coefficient of the product; the exponent of the letter a common to both the terms which we multiply is, $3 + 3$ or 6, and we write after it the letter b, which is found only in the multiplicand, the second partial product is then $-2a^6b$.

The 3rd term $+4a^2b^2$ gives a product affected with the sign $+$, and by the rules applied to the two preceding terms, we find it to be $+4a^5b^2$.

The 2nd line contains the products of all the terms of the multiplicand by the 2nd term $-4a^2b$, of the multiplier; this last having the sign $-$, all the partial products which it gives must have the sign contrary to those of the corresponding terms of the multiplicand; the letters, the coefficients, and the exponents are determined as in the preceding line.

The 3rd line contains the products of all the terms of the multiplicand by the 3rd term $+2b^3$ of the multiplier. This term having the sign $+$ all the products which it gives have the same sign as the corresponding terms of the multiplicand.

After having found all the several products which compose the total, we examine carefully this last to see whether it does not contain similar terms; if it does, they must be reduced according to the rule (19), observing that two terms to be similar should not only contain the same letters but they must also have the same exponents. In the last example there are three reductions, viz:

$$
\begin{array}{lll}
-\ 2a^6b & \text{and } -20a^6b & \text{which gives } -22a^6b \\
+\ 4a^5b^2 & \text{and } +\ 8a^5b^2 & \text{which gives } +12a^5b^2 \\
-16a^4b^3 & \text{and } +10a^4b^3 & \text{which gives } -\ 6a^4b^3.
\end{array}
$$

These reductions being made, we obtain the result as given in the last line of the example.

In order to exercise the student we shall give an example of a multiplication, which it is easy to perform from the preceding rules:

Multiplicand,	$5a^4b^2+7a^3b^3-15a^5c+23b^2d^4-17bc^3d^2-9abcdm^2$
Multiplier,	$11b^3-8c^3+5abc-2bdm.$

$$\text{Partial products,}\left\{\begin{array}{l}55a^4b^5+77a^3b^6-165a^5b^3c+253b^5d^4-187b^4c^3d^2-99ab^4cdm^2.\\-40a^4b^2c^3-56a^3b^3c^3+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2.\\+25a^5b^3c+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2-45a^2b^2c^2dm^2,\\-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3.\end{array}\right.$$

$$\text{Reduced result,}\left\{\begin{array}{l}55a^4b^5+77a^3b^6-140a^5b^3c+253b^5d^4-187b^4c^3d^2-99ab^4cdm^2-40a^4b^2c^3-56a^3b^3c^3.\\+120a^5c^4-184b^2c^3d^4+136bc^6d^2+72abc^4dm^2+35a^4b^4c-75a^6bc^2+115ab^3cd^4-85ab^2c^4d^2.\\-45a^2b^2c^2dm^2-10a^4b^3dm-14a^3b^4dm+30a^5bcdm-46b^3d^5m+34b^2c^3d^3m+18ab^2cd^2m^3.\end{array}\right.$$

XXXIII. By the process of multiplication it is evident, that if all the terms of the multiplicand are of the same degree (27) and those of the multiplier are also of the same degree, all the terms of the product will be of a degree denoted by the sum of the numbers which indicate the degree of the terms of each of the factors.

In the 1st example, (32) the multiplicand has all its terms of the fourth degree and the multiplier of the third; the product has consequently all its terms of the seventh degree.

In the 2nd example, the multiplicand is of the 6th degree, the multiplier of the 3rd degree, the product is therefore of the 9th degree.

Expressions of the kind just referred to, of which all the terms are of the same degree, are called *homogeneous* expressions; the remark made with respect to their products, is useful in preventing any errors which might be committed by forgetting some of the factors in the partial multiplications.

XXXIV. Algebraic operations performed upon literal quantities, in shewing how the different parts of the quantities concur to the formation of the results, lead us frequently to the knowledge of general properties of numbers independent of any systems of notation. The following multiplications, lead us to conclusions of this kind, they are not only worthy of remark, but are of frequent application.

$$\begin{array}{r} a + b \\ a - b \\ \hline a^2 + ab \\ -ab - b^2 \\ \hline a^2 - b^2 \end{array} \qquad \begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

$$\begin{array}{l} a^2 + 2ab + b^2 \\ a + b \\ \hline a^3 + 2a^2b + ab^2 \\ a^2b + 2ab^2 + b^3 \\ \hline a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

It appears from the 1st of these products, that a quantity $a + b$ multiplied by $a - b$ gives $a^2 - b^2$, whence it is evident *that multiplying the sum of two numbers by their difference, the product will be the difference of the squares of these numbers.*

In taking for example the sum 13, of the two numbers 8 and 5, and multiplying it by the difference 3, the product

3×13 or 39, is equal to the difference of 64 (the square of 8), and 25 (the square of 5).

We learn by the 2nd example, in which $a + b$ is twice a factor, *that the second power, or the square of a quantity composed of two parts* a *and* b, *contains the square of the* 1*st part plus twice the product of the* 1*st part by the second, plus the square of the second part.*

The 3rd example in which the 2nd power of $a + b$ is multiplied by the first, shews; that, the 3rd *power or cube of a quantity composed of two parts, contains the cube of the* 1*st part, plus three times the square of the first part multiplied by the 2nd, plus three times the first by the square of the second part, plus the cube of the second part.*

XXXV. In decomposing a quantity into its factors, we dispense with the algebraic operations as long as it can be done, in order to exhibit distinctly the formation of the quantities to be considered; for this reason it is necessary to fix upon proper signs to indicate the multiplication between compound quantities.

With this view, we make use of parenthesis to comprehend the different factors of the product to be indicated. The expression

$$(3\,a^3 - 4\,c^2\,b + b^3)\,(4\,a\,b^2 - a^2\,c - d^2)\,(b^2 - d^2)$$

indicates the product of the compound quantities

$$3\,a^3 - 4\,c^3\,b + b^3;\ 4\,a\,b^2 - a^2\,c - d^2 \text{ and } b^2 - d^2.$$

Some authors, already rather ancient, have drawn lines over the several factors as,

$$\overline{3\,a^2 - 4\,c^2\,b + b^3}\ \overline{4\,a\,b^2 - a^2\,c - d^2}\ \overline{b^4 - d^2}$$

but as these bars might be drawn too long or not sufficiently so, they are liable to great errors; the parenthesis on the contrary never leads into equivocal expressions and has therefore been finally adopted.

Of the Division of Algebraic Quantities.

XXXVI. Algebraic as well as arithmetical division, is to be considered as an operation serving to discover one of the factors of a given product, when the other factor is known. Agreeably to this definition, the quotient (the factor sought) multiplied by the divisor must reproduce the dividend (the product of the two factors).

By applying what is here said, to simple quantities, we shall see by art. (21) that the dividend (called product in multi-

plication) is formed by the factors of the divisor multiplied into those of the quotient, whence *by suppressing in the dividend every factor which composes the divisor, the result will be the quotient sought.*

For example, let it be required to divide the simple quantity $72\,a^5\,b^3\,c^2\,d$ by the simple quantity $9\,a^3\,b\,c^2$; according to the rule above given we must cancel or strike out in the first of these quantities the factors of the second, which are respectively

$$9, a^3, b, \text{ and } c^2.$$

It is necessary, then, in order that the division may be performed, that these factors should be contained in the dividend. Taking them in order, we see that the coefficient 9 of the divisor ought to be a factor of the coefficient 72 of the dividend, or that 9 must be aliquot of 72, which, in fact is the case, as $72 = 9 \times 8$; therefore by cancelling the factor 9, there will remain the factor 8 for the coefficient of the quotient.

It follows also from the rules of multiplication (25), that the exponent 5 of the letter a in the dividend, is the sum of the exponents belonging to that letter in the divisor and quotient; this last exponent therefore will be the difference between the two exponents, viz. $5 - 3 = 2$: thus the letter a in the quotient, has 2 for its exponent. For the same reason, the letter b in the quotient, has an exponent equal to $3 - 1 = 2$. Lastly, the factor c^2 being common to the dividend and divisor must be struck out; consequently we have:

$$8\,a^2\,b^2\,d$$

which is the quotient sought.

As the reasoning of any other example will be the same, we may conclude that, *in order to effect the division of simple quantities, the coefficient of the dividend must be divided by that of the divisors.*

The letters in the dividend which are common to it and the divisor, must be cancelled when they have the same exponent, and when they have a different exponent, the exponent of the divisor must be subtracted from that of the dividend, the remainder is the exponent which the letter must have in the quotient.

Lastly, we must write in the quotient the letters of the dividend which are not in the divisor.

XXXVII. In applying the rule now given for the formation of the exponent of the letters of the quotient, to a letter which has the same exponent in the dividend and the divisor, we should find zero to be the exponent which it ought to have in the quotient; for a^3 dividend by a^3 gives a^0. To ascertain what such an expression can signify, we must go back to its

origin, and consider, that representing the quotient arising from the division of a quantity divided by itself must produce unity, as every quantity is contained once in itself. It follows from this that $\frac{a^3}{a^3}=1$, or a^0, by subtracting the exponent of the divisor, from that of the dividend. Hence it appears, *that the expression* a^0 *is a symbol equivalent to unity, and may consequently be represented by* 1. It is then unnecessary to write such letters as have zero for their exponent, since each represent nothing but unity. Thus $a^2\,b\,c^2$ divided by $a\,b\,c^2$ gives $a^1\,b^0\,c^0$ which is the same as $a^1\times 1\times 1$ or simply a.

The proposition, that; *Every quantity which has zero for its exponent is equal to* 1, is indeed nothing else but the explanation of a result, deduced from the common manner of writing the powers of quantities by exponents.

In order that the division may be performed, it is necessary: 1° that the divisor should contain no letter which is not found in the dividend: 2°, that the exponent of any letter in the divisor should not exceed that of the same letters in the dividend: 3°, that the coefficient of the divisor be aliquot of that of the dividend.

XXXVIII.—When these conditions do not exist, the division can only be indicated in the form of a fraction, agreeably to the manner pointed out (7); after which we must endeavour to simplify the fraction, by cancelling the factors if there are any common both to the dividend and the divisor, for it is evident, that the theory of arithmetical fractions rest, upon principles which are independent of any particular value of their terms, may be applied to fractions represented by letters, as well as to those expressed by numbers.

Agreeably to these principles, *we must in the first place cancel all the numerical factors, common to the coefficients of the dividend and the divisor, and then, the letters which are common to the dividend and divisor, having the same exponent in each. But when the exponent is not the same in each we must subtract the less from the greater, and affix the remainder as the exponent to the letter, which is written only in that term of the fraction which has the highest exponent.*

The following examples, will illustrate this rule:

Be $48\,a^3\,b^5\,c^2\,d$ to be divided by $64\,a^3\,b^3\,c^4\,e$, the quotient can only be indicated in the fractional form;

$$\frac{48.\,a^3\,b^5\,c^2\,d}{64.\,a^3\,b^3\,c^4\,e}:$$

but the coefficient 48 and 64, being both divisible by 16, by

cancelling this common factor, the coefficient of the numerator becomes 3, and that of the denominator 4.

The letter a having the same exponent 3 in both terms of the fraction, it follows that a^3 is a factor common to the dividend and divisor, it may consequently be cancelled.

Proceeding then to the letter b, we must divide the higher power, which is b^5, by b^3 the lower power according to the rule given above, and the quotient b shows that $b^5 = b^3 \times b^2$. Cancelling then the common factor b^3, there will remain in the numerator the factor b^2.

As for the letter c, the factor raised to the higher power c^4 being in the denominator, if we divide it by c^2, we decompose it into $c^2 \times c^2$; and cancelling the factor c^2, common to the two terms, this letter will disappear from the numerator and remain in the denominator with the exponent 2.

Finally, the letters d and e will remain in their respective places, since in the state in which they are they indicate no factor common to the two terms of the fraction.

By these divers operations the proposed fraction is reduced to

$$\frac{3.\,b^2\,d}{4.\,c^2\,e};$$

and this is its most simple expression of the quotient, so long as no numerical values are assigned to the letters; but if numerical values are given to them, then it might be still further reduced by cancelling the common factors as before.

XXXIX. One observation should not be omitted, viz. that if all the factors of the dividend were aliquot of the divisor which besides contain other factors peculiar to it, it is necessary then, after cancelling the factors in the dividend, to substitute unity in the place of the dividend as the numerator of the fraction. In this case indeed, we may suppress in both terms of the fraction all the factors of the numerator, that is, we may divide the two terms of the fraction by the numerator, but this being divided by itself must give unity for the quotient, which becomes the new numerator.

Let there be, for example, the fraction

$$\frac{4\,a^3\,b\,c^2}{12\,a^3\,b^3\,c^2\,d},$$

the factors 12, a^3, b^3, and c^2, of the denominator being respectively divisible by the factors 4, a^3 b and c^2, we can divide both terms of the fraction by the numerator $4\,a^3\,b\,c^2$; but the quantity $4\,a^3\,b\,c^2$ being divided by itself, gives 1 for

the quotient, and the quantity $12\,a^3\,b^3\,c^3\,d$ divided by the first, is reduced to $3\,b^2\,d$: the new fraction is therefore

$$\frac{1}{3\,b^2\,d}.$$

XL. It follows from the rules of multiplication, that if a compound quantity is multiplied by a simple quantity, this last becomes a factor common to all the terms of the first. We avail ourselves of this observation by simplifying the fractions of which the numerator and the denominator are polynomials, having some factors common to all their terms.

Be the expression

$$\frac{6\,a^4 - 3\,a^3\,b\,c + 12\,a^2\,c^2}{9\,a^2\,b - 15\,a^2\,c + 24\,a^3}$$

in examining the quantity $6\,a^4 - 3\,a^2\,b\,c + 12\,a^2\,c^2$ we see that the factor a^2 is common to all the terms, since $a^4 = a^2 \times a^2$ and that moreover the numbers 6, 3 and 12, are all divisible by 3, therefore

$$6\,a^4 - 3\,a^2\,b\,c + 12\,a^2\,c^2 = 2\,a^2 \times 3\,a^2 - b\,c \times 3\,a^2 + 4\,c^2 \times 3\,a^2.$$

The denominator has also $3\,a^2$ as a factor common to all the terms, consequently

$$9\,a^2\,b - 15\,a^2\,c + 24\,a^3 = 3\,b \times 3\,a^2 - 5\,c \times 3\,a^2 + 8\,a \times 3\,a^2$$

cancelling the factor $3\,a^2$, common to all the terms both of the numerator and the denominator, the proposed fraction will be changed into

$$\frac{2\,a^2 - b\,c + 4\,c^2}{3\,b - 5\,c + 8\,a}.$$

XLI. Proceeding now to the case when both the dividend and the divisor are compound, when we can no longer see, at first sight at least, whether the divisor is or is not a factor of the dividend.

As the divisor multiplied by the quotient must reproduce the dividend, this last should consequently contain all the partial products of each term of the divisor, by each term of the quotient; and if we could find again the products arising from each individual term of the divisor, we should only have to divide them by this known term (of the divisor), in the same way as in Arithmetic, where all the figures of the quotient are discovered in dividing successively by the divisor, such numbers of the dividend as are considered the several partial products of this divisor, by the different figures of the quotient. But in numbers these partial products present themselves in order by

beginning with the units placed on the last rank on the left, on account of the subordination established between the units of each figure of the dividend agreeably to the rank which they occupy. This is not the case in Algebra, but we supply the want of such an arrangement by disposing of all the terms, both of the dividend and the divisor, so that the exponents of the powers of the same letter, beginning with the highest, diminish in each term counting from the left to the right hand, as will be seen with reference to the letter a in the quantities

$$5\,a^7 - 22\,a^6\,b + 12\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5$$
$$5\,a^4 - 2\,a^3\,b + 4\,a^2\,b^2$$

of which the 1st is the product, the second the multiplicand in the example given in art. 32: this is termed *arranging* the proposed quantities with reference to one of the letters, of the letter a in the present example.

When they are so arranged, it is evident, that whatever be the factor by which it is necessary to multiply the 2nd quantity in order to obtain the first, the 1st term $5\,a^7$, results from the multiplication of the 1st term $5\,a^4$ of the 2nd quantity by the 1st term of the factor sought, in which a has the highest exponent, and which takes the first place in this factor when the terms of it are arranged with reference to the letter a. By dividing then the simple quantity $5\,a^7$ by the simple quantity $5\,a^4$, the quotient a^3 will be the first term of the factor sought. But by the rules of multiplication, the total product ought to contain the divers partial products resulting from the multiplication of the whole multiplicand by each term of the multiplier; it follows that the quantity considered here as dividend ought to contain the products of all the terms of the divisor, $5\,a^4 - 2\,a^3\,b + 4\,a^2\,b^2$ by the 1st term of the quotient a^3; and consequently, if we subtract from the dividend these products, which are

$$5\,a^7 - 2\,a^6\,b + 4\,a^5\,b^2$$

the remainder $- 20\,a^6\,b + 8\,a^5\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5$, will contain only those, which result from the muliplication of the divisor, by the 2nd, 3rd, 4th, &c. term of the quotient.

Considering the remainder as a new dividend, its first term—$20\,a^6\,b$ in which a has the highest exponent, can only be obtained by the multiplication of the 1st term of the divisor by the 2nd term of the quotient. But the 1st term of this partial dividend having the sign — before it, we must ascertain that which is to be prefixed to the corresponding term of the quotient; this may easily be done by the last

rule of art. 31, for the quantity $-20\ a^6\ b$, regarded as a partial product, having a sign contrary to that of the multiplicand $5\ a^4$, it follows that the multiplier must necessarily have the sign —. The division being then performed upon the simple quantities $-20\ a^6\ b$ and $5\ a^4$ gives $-4\ a^2\ b$ for the 2nd term of the quotient. If this last term be multiplied by all the terms of the divisor, and the products subtracted from the partial dividend, the remainder $10\ a^4\ b^3 - 4\ a^3\ b^4 + 8\ a^2\ b^5$ can only contain the products of the divisor by the 3rd, &c. terms of the quotient.

Regarding this remainder as a new dividend, its first term $10\ a^4\ b^3$ must be the product of the 1st term of the divisor by the 3rd of the quotient; consequently, this last is obtained by dividing the simple quantities $10\ a^4\ b^3$ and $5\ a^4$ the one by the other. The quotient $2b^3$, being multiplied by all the terms of the divisor, gives products, which being subtracted from this last dividend, leaves no remainder, and shews that the quotient has only three terms.

If the nature of the question had been such as to require a greater number of terms, they might evidently have been found like the preceding, and if, as we have supposed, the dividend contained the divisor for a factor, the subtraction of the product of the divisor by the last term of the quotient, must always exhaust this last partial dividend.

XLII. In order to facilitate the practice of the rules found above:—

1°. *The dividend as well as the divisor are disposed in the same way as for the division of numbers, by arranging them with reference to some letter; viz. by writing their terms in the order of the exponents of this letter, beginning with the highest.*

2°. *We divide the first term of the dividend by the* 1st *term of the divisor, and write the result in the place of the quotient.*

3°. *We multiply all the terms of the divisor by the term of the quotient just found, subtract it from the dividend, and then we reduce the similar terms.*

4°. *We consider this remainder as a new dividend, of which we divide the first term by the first term of the divisor, and write the result as the second term of the quotient, and continue the operation until all the terms of the dividend are exhausted.*

Observing that when a product has the same sign as the multiplicand, the multiplier has the sign +, and that when a product has a sign contrary to that of the multiplicand, the multiplier has the sign — (31), we conclude; *that when the partial dividend and the first term of the divisor have the same sign, the quotient must have the sign* +; *but if they have*

contrary signs, the quotient must have the sign — ; this is the rule for the signs.

The partial divisions are performed by the rules given for the division of simple quantities.

We write in the quotient the letters common to the dividend and the divisor, with an exponent equal to the difference of the exponents of these letters in the two terms, and finally, such letters as are in the dividend only; these are the rules for the letters and exponents.

XLIII. Applying these rules to the quantities

$$5\,a^7 - 22\,a^6\,b + 12\,a^6\,b^2 - 6\,a^4\,b^3 - 4\,a^3\,b^4 + 8\,a^2\,b^5$$

$$5\,a^4 - 2\,a^3\,b + 4\,a^2\,b^2.$$

which have been employed as an example above, we arrange them as we place the dividend and the divisor in arithmetic.

Divisor.	Dividend.	Quotient.
$5a^4-2a^3b+4a^2b^2$	$5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	$a^3-4a^2b+2b^3$
	$-5a^7+2a^6b-4a^5b^2$	
1st Remainder,	$-20a^6b+8a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$	
	$+20a^6b-8a^5b^2+16a^4b^3$	
2nd Remainder,	$+10a^4b^3-4a^3b^4+8a^2b^5$	
	$-10a^4b^3+4a^3b^4-8a^2b^5$	
3rd Remainder,	$0 \quad 0 \quad 0$	

The sign of the 1st term of the dividend $5a^7$ being the same as that of the 1st term $5a^4$ of the divisor, the quotient must have the sign $+$; but as this partial quotient forms the first term of the total quotient, the sign is omitted.

By dividing $5a^7$ by $5a^4$ the result a^3 is written in the quotient. Multiplying successively the three terms of the divisor, by the 1st term a^3, of the quotient, and writing the products under the corresponding terms of the dividend, and after having changed the signs of these products, to denote subtraction, we have the quantity

$$-5a^7 + 2a^6b - 4a^5b^2,$$

making then with the dividend the reduction, we obtain for a remainder,

$$-20a^6b + 8a^5b^2 - 6a^4b^3 - 4a^3b^4 + 8a^2b^5.$$

In continuing the division with this remainder, the 1st term $-20a^6b$, divided by $5a^4$, gives $-4a^2b$ for the quotient, with the sign $-$ before it, the divisor and dividend having contrary signs. Multiplying this quotient $-4a^2b$, by all the terms of the divisor, and changing the signs, we form the following quantity:

$$20a^6b - 8a^5b^2 + 16a^4b^3$$

which, taken with the dividend and reduced, gives for a remainder

$$10a^4b^3 - 4a^3b^4 + 8a^2b^5.$$

Dividing the 1st term of this new dividend, $10a^4b^3$, by the 1st term $5a^4$ of the dividend, and multiplying the result $+2b^3$ by the whole divisor, placing the products under the partial dividend, observing to change the signs, then making the reduction, there remains nothing, shewing that $+2b^3$ is the last term of the quotient sought, the expression of which is

$$a^3 - 4a^2b + 2b^3.$$

XLIV. It is proper to observe here, that in division, the multiplication of the different terms of the quotient by the divisor, often produces terms that are not to be found in the dividend, and which must afterwards be divided by the 1st term of the divisor. These are the same terms that cancel each other, since the dividend has been formed by the multiplication of the two factors, the divisor and the quotient. The following is a remarkable example of these reductions:

Be $a^3 - b^3$ to be divided by $a - b$

Division.

$$
\begin{array}{r|l|l}
a-b & a^3-b^3 & a^2+ab+b^2 \\
 & -a^3+a^2b & \\
\hline
 & +a^2b-b^3 & \\
 & -a^2b+ab^2 & \\
\hline
 & +ab^2-b^3 & \\
 & -ab^2+b^3 & \\
\hline
 & 0 &
\end{array}
$$

Multiplication.

$$
\begin{array}{l}
a-b \\
a^2+ab+b^2 \\
\hline
a^3-a^2b \\
\quad +a^2b-ab^2 \\
\qquad +ab^2-b^3 \\
\hline
\text{Result } a^3 - \qquad b^3
\end{array}
$$

The 1st term a^3 of the dividend, divided by the 1st term a of the divisor, gives for the quotient a^2; multiplying this quotient by the divisor, and changing the signs of the products, we find $- a^3 + a^2 b$; the first term $- a^3$, destroys the first term of the dividend; but there remains the term $a^2 b$, which is not found at first in the dividend. But as this term contains the letter a, it can be divided by the 1st term of the divisor; we thus obtain $+ a b$. Multiplying this quotient by the divisor, and changing the signs of the products, we get $- a^2 b + a b^2$; the term $- a^2 b$ of the products cancels the 1st term of the dividend, but there remains the term $+ a b^2$ which is not in the dividend. Dividing it by a, it gives for the quotient $+ b^2$; multiplying this partial quotient, by the divisor, we get, after having changed the signs of the products, $- a b^2 + b^3$: the 1st term $- a b^2$ destroys the first term of the dividend, and the 2nd $+ b^3$ cancels the last term $- b^3$.

The mechanical part of the operation will be better understood, if we look for a moment at the multiplication of the quotient $a^2 + a b + b^2$ by the divisor $a - b$, placed in juxta-position of the division; we see that all the terms reproduced in the process of dividing, are those which destroy each other in the result of the multiplication.

XLV. It happens sometimes that the quantity, with reference to which the arrangement is made, has the same power in several terms both of the dividend and divisor. In this case the terms should be placed in the same column, that is written immediately the one under the other, observing to arrange the remaining ones with reference to another letter. Let there be

$$-a^4 b^2 + b^2 c^4 - a^2 c^4 - a^6 + 2 a^4 c^2 + b^6 + 2 b^4 c^2 + a^2 b^4$$

to be divided by $a^2 - b^2 - c^2$.

Arranging the 1st of these quantities with reference to the letter a, the terms $- a^4 b^2$ and $2 a^4 c^2$, should then be placed in the same column, in another, the terms $+ a^2 b^4$ and $- a^2 c^4$

and in the last column, the three terms $+ b^6$, $+ 2\,b^4\,c^2$ and $+ b^2\,c^4$ disposing them with reference to the letter b, as may be seen in the next page.

The 1st term $-a^6$ of the dividend being divided by the 1st term a^2 of the divisor, gives $-a^4$ for the first term of the quotient; forming then the products of this quotient by all the terms of the divisor, and changing the signs of their products in order to subtract them from the dividend, and writing in the same column the terms containing the same power of a, we obtain, after the reduction of similar terms, the 1st remainder, which is now to be considered as the 2nd partial dividend.

The first term $-2\,a^4\,b^2$ of this new dividend being divided by a^2, gives the second term of the quotient $-a^2\,b^2$, then forming the products of this quotient by all the terms of the divisor, changing their signs to indicate their subtraction from the partial dividend, and placing in the same column, the terms containing the same power of a, we get, after the reduction of similar terms, the 2nd remainder, which we take for the 3rd partial dividend.

The operation being continued in the same manner with this 2nd remainder and the following ones, we shall have three terms in the quotient; the last of which being multiplied by all the terms of the divisor, furnishes products which, being subtracted from the 4th remainder, exhaust it entirely. The division being thus exactly performed, it follows that the divisor is a factor of the dividend.

Changing the signs of the products of the quotient by the divisor, and then adding them to the dividend, amounts indeed to the same thing as leaving their signs unchanged and subtracting them from the dividend, but the process here adopted makes in many cases the operation more clear and easy.

Divisor.	*Dividend.*	*Quotient.*
$a^2-b^2-c^2$	$\begin{array}{l} -a^6-a^4b^2+a^2b^4+b^6 \\ \quad +2a^4c^2-a^2c^4+2b^4c^2 \\ \quad\quad +b^2c^4 \\ +a^6-a^4b^2 \\ \quad -a^4c^2 \end{array}$	$\begin{array}{l} -a^4-2a^2b^2-b^4 \\ \quad +a^2c^2-b^2c^2 \end{array}$

1st Remainder,
$$\begin{array}{l} -2a^4b^2+a^2b^4+b^6 \\ +a^4c^2-a^2c^4+2b^4c^2 \\ \qquad +b^2c^4 \\ +2a^4b^2-2a^2b^4 \\ \qquad -2a^2b^2c^2 \end{array}$$

2nd Remainder,
$$\begin{array}{l} +a^4c^2-a^2b^4+b^6 \\ \qquad -2a^2b^2c^2+2b^4c^2 \\ \qquad -a^2c^4+b^2c^4 \\ -a^4c^2+a^2b^2c^2 \\ \qquad +a^2c^4 \end{array}$$

3rd Remainder,
$$\begin{array}{l} -a^2b^4+b^6 \\ -2a^2b^2c^2+2b^4c^2 \\ \qquad +b^2c^4 \\ +a^2b^4-b^6 \\ \qquad -b^4c^2 \end{array}$$

4th Remainder,
$$\begin{array}{l} -2a^2b^2c^2+b^4c^2 \\ \qquad +b^2c^4 \\ +2a^2b^2c^2-b^4c^2 \\ \qquad -b^2c^4 \end{array}$$

No Remainder, $\quad 0$

XLVIII. The form under which a quantity appears will sometimes immediately suggest the factors into which it may be decomposed. Suppose for example, we have,

$$8\,a^6-4\,a^3\,b^2+4\,a^3+2\,a^3-b^2+1$$

to be divided by $2\,a^3-b^2+1$; this divisor forming the three last terms of the dividend, it is only necessary to see if it is a factor of the three first terms, but these have evidently $4\,a^3$ for a common factor, for

$$8\,a^6-4\,a^3\,b^2+4\,a^3=4\,a^3\,(2\,a^3-b^2+1).$$

The dividend thus may be represented by

$$4\,a^3\,(2\,a^3-b^2+1)+2\,a^3-b^2+1$$

or
$$(2\,a^3-b^2+1)\,(4\,a^3+1)$$

the division then is done by inspection, by cancelling the factor $2a^3-b^3+1$, common to the dividend and the divisor, and the quotient will be:

$$4a^3+1.$$

After some practice, methods of this kind will readily occur by which algebraic operations are abridged.

By frequent exercise in examples of this kind, the student will easily learn to resolve a quantity into its factors. These resolutions are often rendered evident by indicating only, instead of performing the operations that present themselves.

Examples in Division or Multiplication of Compound Quantities.

Divisor and Quotient, or Multiplicand and Multiplier. *Dividend, or Product.*

1. $(5xy-ab+z)4a = \ldots\ldots\ldots\ldots\ 20axy-4a^2b+4az.$
2. $(-(2+\frac{1}{2})z+x)\ (-16az) = \ldots\ldots\ldots\ 40az^2-16axz.$
3. $(3a^2+2x^3-3y^2)3xy = \ldots\ldots\ldots\ 9a^2xy+6x^4y-9xy^3.$
4. $(x^2+xy+y^2)(x-y) = \ldots\ldots\ldots\ x^3-y^3.$
5. $(y^4-y^3+y^2-y+1)\ (y+1) = \ldots\ldots\ldots\ y^5+1.$
6. $(x^2-ax+c)\ (x+d) = x^3-(a-d)x^2-(ad-c)x+cd.$
7. $(x^7-x^2y^2+x^5y^2)(\ x^3+1) = \ldots\ x^2(x^8-x^6y^2+x^5-y^2).$
8. $(x^5-y^5+y^7)\ (x^2+y^2+1) = x^2(x^5+x^4y^2-x^3-y^5+y^7)-y^5+y^9$
9. $[-x^3(2x^3-3y^3)+\frac{a^4}{4}-\frac{3}{2}a^2y^3](a^3y-2ax^3y) = (\frac{a^2}{2}-x^3$

being a common factor) $(\frac{a^2}{2}-x^3)\ [(2x^3+a^2-3y^3)\times 2ay]$

$= ay\ (6x^3y^3-4x^6+a^4-3ay^3).$

10. $(2x^2+3ax-a^2)\ (2x^2-3ax+a^2) = 4x^4-9a^2x^2+6a^3x-a^4.$
11. $(x-z+\frac{y}{x+z})\ (x+z) = \ldots\ldots\ldots\ x^2-z^2+y.$
12. $(x^4+x^3y+x^2y^2+xy^3+y^4)\ (x-y) = \ldots\ldots\ldots x^5-y^5.$
13. $(x^3-x^2y+xy^2-y^3)\ (x+y) = \ldots\ldots\ldots x^4-y^4.$
14. $(x^4-x^3y+x^2y^2-xy^3+y^4)\ (x+y) = \ldots\ldots\ldots x^5+y^5.$
15. $(x^3+x^2y+xy^2+y^3)\ (x-y) = \ldots\ldots\ldots x^4-y^4.$

Hence we may conclude, that x^n+y^n is divisible by the sum of the roots $x+y$, and x^n-y^n by their difference when the exponent is an odd number. But x^n-y^n is divisible by the sum and also by the difference when n is an even number.

16. $(2x^2+5x-7)(2x-3) = \ldots \quad 4x^3+4x^2-29x+21.$

17. $(6a-3mb-5c)(6a-3mb-5c)$ or $(6a-3mb-5c)^2 =$
$36a^2+9m^2b^2+25c^2-36abm-60ac+30bcm.$

18. $(4x^2-3y+7a)(4x^2+3y-7a)=16x^4-42ay-49a^2-9y^2.$

19. $(3x+2y)^2(3x-2y)^3 = 243x^5-162x^4y-216x^3y^2+144x^2y^3$
$+48xy^4-32y^5.$

20. $(5a^4-4b^3+3c^2)(5x^2-1)=25a^4x^2-5a^4-20b^3x^2-4b^3$
$+15c^2x^2-3c^2.$

21. $(x^4+8x^2+4)(x^2-16) = x^6-8x^4-124x^2-64.$

22. $(3a-2b+8c)(6a+15b-2c-4d) = 18a^2+33ab+42ac$
$-12ad-30b^2+124bc+8bd-16c^2-32cd.$

23. $(15ax-3a-10bx+5mx-2b-m)(5bx-b) = \ldots\ldots\ldots\ldots$
$b(75ax^2-3ax+3a-50bx^2-25mx^2-10mx+2b+m).$

24. $(11x-11y+11\frac{a}{2})(2ay-2ax) = 11ax(4y-2x-a)$
$+11ay(a-2y).$

25. $(4a^2x^2-yb^6+x^4)(5x^2-20a^2) = 20a^2(b^6y-4a^2x^2)-$
$5x^2(b^6y-x^4).$

26. $(\frac{13ab}{5c}-\frac{6b^2}{2})(10c^2b+5ac) = ab(11bc+13a)-30c^2b^3.$

27. $\frac{18a^{-5}b^3}{7c^{-2}d^{-6}} \times \frac{4a^6b^{-5}}{9c^3d^9} = \ldots\ldots\ldots\ldots\ldots\ldots \frac{8a}{7b^2cd^3}.$

28. $\frac{1}{3a^{-3}b^{-m}cd} \times \frac{1}{5a^{-m}b^2} = \ldots\ldots\ldots\ldots\ldots\ldots \frac{a^{m+3}b^{m-2}}{15cd}.$

29. $(a^m+b^p-2c^n)(2a^m-3b) = 2a^{2m}+2a^mb^p-4a^mc^n$
$-3a^mb-3b^{p+1}+6bc^n.$

30. $(2a^{3-2m}b^{n+3}+3a^{m+1}b^{n+2}+c^p)(a^{m-1}b^{1-2m}-ca^p)$
$=2a^{2-m}b^{n-2m+4}+3a^{2m}b^{n-2m+3}+a^{m-1}b^{1-2m}c^p-2a^{p-2m+3}b^{n+3}c- \ldots$
$3a^{p+m+1}b^{n+2}c-a^pc^{p+1}.$

31. $(x^{-3p}+3a^mx^{-2p}-10a^{2m}x^{-p})(a^2x^q+5a^{m+2}x^{q+p}-2a^{2m+2}x^{q+2p})$
$=a^2x^{q-3p}+8a^{m+2}x^{q-2p}+3a^{2m+2}x^{q-p}-56a^{3m+2}x^q+20a^{4m+2}x^{p+q}.$

32. $(3a^{4-3m}bc^{m-2}+17a^{-2}b^{m+1})(3a^{6m-2}b^{2m}c^{3-4m}-8)$
$=9a^{3m+2}b^{2m+1}c^{1-3m}+51^{6m-4}b^{3m+1}c^{3-4m}-24a^{4-3m}bc^{m-2} - \ldots\ldots$
$136a^{-2}b^{m+1}.$

33. $(2ab^7-5a^4b^3-17a^7b)(\frac{3}{2}a^3b^5+\frac{1}{2}a^6b) = 3a^4b^{12}-7a^7b^8$

$$-\frac{51}{2}a^{10}b^{8}-\frac{5}{4}a^{10}b^{4}-\frac{17}{4}a^{13}b^{2}=\frac{a^{4}b^{2}}{4}(12b^{10}-28a^{3}b^{6}-102a^{6}b^{4}$$
$$-5a^{8}b^{2}-17a^{9}.)$$

34. $(5a^{3}b^{2}c^{3}-2a^{2}b^{2}c^{4}-3ab^{2}c^{5}-7bc^{6})\ (a^{2}bc^{2}-4abc^{3})=\ldots\ldots\ldots$
$5a^{5}b^{3}c^{5}-22a^{4}b^{3}c^{6}+5a^{3}b^{3}c^{7}+12a^{2}b^{3}c^{8}-7a^{2}b^{2}c^{8}+28ab^{2}c^{9}=\ldots\ldots\ldots$
$ab^{2}c^{5}(5a^{4}b-22a^{3}bc+5a^{2}bc^{2}+12abc^{3}-7ac^{3}+28c^{4}.)$

35. $$\left(\frac{2a^{3}b^{2}}{5}-\frac{3}{4}a^{2}b^{3}+6ab^{4}\right)\left(\frac{a^{4}}{2}-2a^{3}b\right)=\frac{a^{7}b^{2}}{5}-\frac{47}{40}a^{6}b^{3}$$
$$+\frac{9}{2}a^{5}b^{4}-12a^{4}b^{5}=\frac{a^{4}b^{2}}{40}(8a^{3}-47a^{2}b+180ab^{2}-480b^{3}.)$$

36. $(2a^{-3}x^{3}-3ax^{4})\ (-a^{-5}x^{2}+7a^{-1}x^{3}+8a^{3}x^{4},)=\ldots-2a^{-8}x^{5}$
$+17a^{-4}x^{6}-5x^{7}-24a^{4}x^{8}=x^{5}(-2a^{-8}+17a^{-4}x-5x^{2}-24a^{4}x^{3}.)$

37. $$\left(\frac{a^{2}c}{b^{2}}+\frac{2a^{3}c}{b}-a^{4}c\right)\left(\frac{a}{b^{3}}+\frac{3a^{2}}{b^{2}}+c^{2}\right)=\frac{a^{3}c}{b^{5}}+\frac{a^{4}c}{b^{4}}$$
$$-\frac{7a^{5}c}{b^{3}}-\frac{3a^{6}c}{b^{2}}+\frac{a^{2}c^{3}}{b^{4}}-\frac{2a^{3}c^{3}}{b}-4a^{4}c^{3}.$$

38. $(a^{2}d^{2}-2acd^{2}+c^{2}d^{2}+ac^{2}d)(ad-cd)=a^{3}d^{3}-3a^{2}cd^{3}+.$
$3ac^{2}d^{3}-c^{3}d^{3}+a^{2}c^{2}d^{2}-ac^{3}d^{2}=d^{2}(a^{3}d-3a^{2}cd+3ac^{2}d-c^{3}d$
$+a^{2}c^{2}-ac^{3}.)$

39. $(a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^{2}c^{n}+b^{p}c^{m})\ \ (a^{-n}b^{-p-1}+bc^{n-1})=\ldots$
$a^{3m-n}b^{2p}c-a^{2m+n-1}b^{1-p}c^{n}+a^{-n}b^{-1}c^{m}+a^{3m-n}b^{3p+2}c^{n}-\ldots\ldots\ldots\ldots$
$a^{2m+n-1}b^{3}c^{2n-1}+b^{p+1}c^{m+n-1}.$

40. $(a^{m}+3a^{m-1}b^{n}-6a^{m-2}b^{2n})\ \ (a^{n}b^{n}-7a^{n-1}b^{2n})=a^{m+n}b^{n}-$
$4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n}.$

41. $$\frac{2x^{3n-5m}y^{2n-3}}{7a^{m}b^{3}c}\cdot\frac{3ab^{n-1}y^{5}}{4x^{1-5m}}=\frac{3b^{n-4}y^{2n+2}x^{3n-1}}{14a^{m-1}c}.$$

42. $$\frac{2c^{3}(1+z^{2})^{2}}{d^{7}z^{9}}\cdot\frac{7d^{9}z^{5}}{14c^{8}f^{3}(1+z^{2})^{-6}}=\frac{d^{2}(1+2^{2})^{8}}{c^{5}f^{3}z^{4}}.$$

43. $$\frac{-5c^{2}a^{-m}b^{n}}{6}\cdot\frac{8}{ca^{p}b^{-q}cd^{5}}=-\frac{5}{3}\cdot\frac{cb^{n+q}}{a^{p+m}d^{5}}.$$

44. $$\frac{42a^{2m-n-1}b^{-m-2}(n+m)^{-2}}{9a^{2n+1}b^{m+3}(n+m)}\ .\ \frac{3a^{n}b^{3}}{7(n+m)^{-3}}=\frac{2a^{2m-2n-2}}{b^{2m+2}}.$$

Of Algebraic Fractions.

XLVI. By applying the process of algebraic division to two quantities of which the one is not a factor of the other, the impossibility of performing it, becomes evident, since we arrive in the course of the operation at a remainder, of which the first term is not divisible by that of the divisor. To give an example:

$$a^2+b^2 \qquad a^3 + a^2b + 2b^3 \qquad a+b$$
$$ab^2$$

1st Remainder, $\qquad a^2b - ab^2 + 2b^3$
$$-a^2b - b^3$$

2nd Remainder, $\qquad - ab^2+ b^3$

The first term, $- a b^2$, of this 2nd remainder cannot be divided by a^2, the 1st term of the divisor; so that the process of division is arrested at this point. We might, however, as in Arithmetic, add to the quotient $a + b$, the fraction

$$\frac{- ab^2+b^3}{a^2+b^2},$$

having the remainder for the numerator, and the divisor for the denominator; the quotient will consequently have the expression,

$$a + b + \frac{b^3 - ab^2}{a^2 + b^2}.$$

It is evident that *the division must cease when we arrive at a remainder, of which the first term contains the letter with reference to which the terms are arranged in an inferior power to that of the same letter in the 1st term of the divisor.*

XLVII. When the algebraic division of two quantities can not be performed, the expression of the quotient remains indicated under a fractional form, having the dividend as numerator and the divisor for denominator; to carry it to the highest degree of simplicity, we must see if the dividend and divisor contain some common factors, which might then be cancelled. But when the terms of the fraction are polynomials, the common factors are not so easily discovered as when they are simple quantities, they are generally found by a method analogous to that which is given in arithmetic for finding *the greatest common divisor* of two proposed quantities.

As no relative magnitudes of algebraic expressions can be assigned so long as we do not give values to the letters which

they contain, the denomination of *the greatest common divisor* therefore, applied to these expressions, ought not to be taken altogether in the same sense as in arithmetic.

In algebra, we are to understand by *the greatest common divisor* of two expressions, that which contains the greatest number of factors in all its terms, or that which is of the highest degree, (27.) Its determination rests, as in arithmetic, upon this principle: *Every divisor common to two quantities, must divide the remainder after their division.*

The demonstration given in arithmetic is rendered clearer by employing algebraic symbols. Indeed, let A and B be the two proposed quantities, D their common divisor, Q the quotient of the division of A by B, and R the remainder, we have:

$$A = BQ+R\,;$$

dividing then both members of the equation by the common divisor D, we get

$$\frac{A}{D} = \frac{BQ}{D} + \frac{R}{D};$$

and making $\frac{A}{D} = a$, $\frac{B}{D} = b$, quotients which are exact by hypothesis, the above equation will be changed into

$$a = bQ + \frac{R}{D}, \text{ whence } a - bQ = \frac{R}{D}:$$

but since the first member, which in this case is composed of the same terms as the second, is now a whole number, R must necessarily be divisible by D.

Reciprocally, every divisor common to the quantities B and R must divide A; for making $\frac{B}{D} = b$ and $\frac{R}{D} = r$, the equation $\frac{A}{D} = \frac{BQ}{D} + \frac{R}{D}$

becomes $$\frac{A}{D} = bQ + r:$$

thence it follows that A is necessarily divisible by D if b and r are whole numbers.

Agreeably to these principles, we *begin*, as in Arithmetic, *by trying whether one of the quantities is not itself the divisor of the other; if the division cannot be exactly performed, we divide the first divisor by the remainder and so on; and that remainder, which will exactly divide the preceding, will be the greatest common divisor of the two proposed quantities.* It will however be necessary, in the divisions indicated, to have regard to what appertains to the nature of algebraic quantities.

We are not, in the first place, to seek a divisor common to the two algebraic expressions, unless they contain common letters; and we must select from them a letter, with reference to which the proposed expressions are to be arranged, and take as dividend that expression in which this letter has the highest exponent, the other being the divisor.

Let there be the two quantities

$$3a^3 - 3a^2b + ab^2 - b^3$$
$$4a^2b - 5ab^2 + b^3$$

which are already arranged with reference to the letter a; the 1st is taken for the dividend and the 2nd for the divisor. From the very beginning a difficulty presents itself which we never meet with in numbers, and this is, that the 1st term of the divisor will not exactly divide the 1st term of the dividend, on account of the factors 4 and b in the one, which are not to be found in the other. But the letter b being common to all the terms of the divisor, but not to those of the dividend, it follows (40) that b is a factor of the divisor, and that it is not of the dividend. Now every divisor common to two quantities, can only be composed of factors which are common to both; if then there be such a divisor with respect to the two quantities proposed, it is to be looked for amongst the factors of the quantity $4a^2 - 5ab + b^2$, which remains after having cancelled b of the quantity $4a^2b - 5ab^2 + b^3$, so that the question reduces itself to finding the greatest common divisor of the two quantities,

$$3a^3 - 3a^2b + ab^2 - b^3$$
$$4a^2 - 5ab + b^2.$$

For the same reason that we may cancel in one of the proposed quantities the factor b, which is not contained in the other, we may likewise introduce in this last a new factor, provided it be not a factor of the first. By this step, the greatest common divisor of these quantities, which is composed of those factors only which are common to both, will not be affected. Taking advantage of this principle, we must multiply the quantity $3a^3 - 3a^2b + ab^2 - b^3$ by 4, which is not a factor of the quantity $4a^2 - 5ab + b^2$, in order to render the 1st term of the one divisible by the 1st term of the other.

In this manner we shall get as dividend, the quantity

$$12a^3 - 12a^2b + 4ab^2 - 4b^3,$$

and for divisor the quantity

$$4a^2 - 5ab + b^2,$$

and the partial quotient will be $3a$.

Multiplying the divisor by this quotient, and subtracting the product from the dividend, we shall get for remainder

$$3a^2b + ab^2 - 4b^3$$

a quantity which, according to the principle stated at the commencement of the present article, must have with $4a^2 - 5ab + b^2$, the same common divisor as the first.

Availing ourselves of the remark made above, we suppress the factor b, common to all the terms of this last remainder, and multiply it by 4, in order to render the first term divisible by that of the divisor.

The dividend will then be,

$$12a^2 + 4ab - 16b^2,$$

and the divisor, the quantity

$$4a^2 - 5ab + b^2,$$

and the quotient thence arising is 3.

Multiplying the divisor by this quotient, and subtracting the product from the dividend, we get the remainder,

$$19ab - 19b^2;$$

the question is now reduced to finding the greatest common divisor to this quantity, and

$$4a^2 - 5ab + b^2.$$

But the letter a, with reference to which the division is made, being contained in the remainder in the 1st degree only, while it is of the second degree in this divisor, it is this we must take for the dividend and the remainder for the divisor.

Before beginning this new division, we expunge the factor $19\,b$ common to both the terms $19\,ab - 19\,b^2$ of the divisor, and not being a factor of the dividend, we have then for dividend the quantity

$$4a^2 - 5ab + b^2,$$

and for a divisor $\quad a - b.$

The division leaving no remainder, shews that $a - b$ is the greatest common divisor required.

By retracing these steps, it may be proved *a posteriori*, that the quantity $a - b$ must exactly divide the two proposed quantities, and that it is the most compounded quantity of those that will divide the proposed quantities. Dividing then the two proposed quantities by $a - b$, we obtain,

$$3a^3 - 3a^2b + ab^2 - b^3, \text{ and } 4a^2b - 5ab^2 + b^3,$$

which can be resolved as follows;

$$(3a^2 + b^2)\,(a - b), \text{ and } (4ab - b^2)\,(a - b)$$

XLIX. When the quantity, which is taken for a divisor, has several terms containing the letter, with reference to which the arrangement is made, of the same degree, great attention is required, without which the operation would not terminate. For example:

Be the given quantities;

$$a^2b + ac^2 - d^3, \text{ and } ab - ac + d^2.$$

Preparing the operation as for a common division:

$$\begin{array}{r|l} ab - ac + d^2 & a^2b + ac^2 - d^3 \\ & -a^2b + a^2c - ad^2 \\ \hline \text{Remainder,} & a^2c + ac^2 - ad^2 - d^3 \end{array}$$

by dividing, first, a^2b by ab, we find for the quotient a; multiplying the divisor by this quotient, and subtracting the products from the dividend, the remainder will contain a new term, in which a will be of the second degree, viz. a^2c, arising from the product of $-ac$ by a. Thus no progress has been made; for by taking the remainder,

$$a^2c + ac^2 - ad^2 - d^3$$

for a dividend, and multiplying by b, to render the division by ab possible, we have

$$\begin{array}{r|l|l} ab - ac + d^2 & a^2bc + abc^2 - abd^2 - bd^3 & ac \\ & -a^2bc + a^2c^2 - acd^2 & \\ \hline \text{Remainder,} & a^2c^2 + abc^2 - acd^2 - abd^2 - bd^3 & \end{array}$$

and the term $-ac$ produces still a term a^2c^2, in which a is of the second degree.

To avoid this inconvenience, we must observe, that the divisor $ab - ac + d^2 = a(b - c) + d^2$, by uniting the terms $ab - ac$ into one; and, for the sake of simplifying the calculations, we make $b - c = m$; the divisor will be $am + d^2$; but then we must multiply the whole dividend $a^2b + ac^2 - d^3$ by the factor m, in order to obtain a new dividend, of which the first term be divisible by the quantity am, forming the 1st term of the divisor: the operation then becomes

$$\begin{array}{r|l|l} am + d^2 & a^2bm + ac^2m - d^3m & ab + c^2 \\ & -a^2bm - abd^2 & \\ \hline \text{1st Remainder,} & -abd^2 + ac^2m - d^3m & \\ & -ac^2m - c^2d^2 & \\ \hline \text{2nd Remainder,} & -abd^2 - c^2d^2 - d^3m. & \end{array}$$

The terms involving a^2 now disappear from the dividend, and there only remain the terms which have the first power of a. To make these disappear, we divide the term ac^2m by am, and we get for a quotient c^2; multiplying the divisor by this

quotient, and subtracting the products from the dividend, we obtain the 2nd remainder. Taking this second remainder for a new dividend, the factor d^2, which is not a factor of the divisor, can then be cancelled, we then have

$$-ab-c^2-dm,$$

which, being multiplied anew by m, becomes

$$\begin{array}{r|l|l} am+d^2 & -abm-c^2m-dm^2 & -b \\ & +abm+bd^2 & \\ \hline \text{3rd Remainder,} & bd^2-c^2m-dm^2. & \end{array}$$

The remainder $bd^2—c^2m—dm^2$ of this last division, containing a no longer, it follows, that if in the proposed quantities there exist a common divisor, that divisor must be independent of the letter a.

Arrived at this point, the division can no longer be continued with reference to the letter a; but observing that if there be a common divisor, independent of a, to the two quantities $bd^2-c^2m-dm^2$ and $am+d^2$, it must divide separately the two parts am and d^2 of the divisor; for in general, if a quantity is arranged with reference to the powers of the letter a, every divisor of this quantity, independent of a, must divide separately the quantities multiplied by the different powers of this letter.

To be convinced of the truth of this observatian, it is sufficient to observe, that in this case each of the quantities proposed must be the product of a quantity dependent on a and of the common divisor, which does not depend on it. Now if we have for example, the expression

$$Aa^4+Ba^3+Ca^2+Da+E,$$

in which the letters A, B, C, D, E designate any quantity whatever, independent of a, and if it be multiplied by a quantity M, also independent of a, the product

$$MAa^4+MBa^3+MCa^2+MDa+ME$$

arranged with reference to the letter a, will contain still the same powers of a as before; but the coefficient of each of these powers will be a multiple of M.

This being premised, if we restore for m its value $b—c$, which this letter represents, we have the quantities

$$bd^2-c^2(b-c)-d(b-c)^2 \text{ and } a(b-c)+d^2;$$

and it is evident, that $b-c$ and d^2 have no common factor; the two proposed quantities then have no common divisor. Were it not evident by the mere inspection that there is no common divisor between $b-c$ and d^2, it would be necessary to seek their greatest common divisor, by arranging the several terms with

reference to the same letter, and then to see if it would not also divide the quantity

$$bd^2 - c^2 (b - c) - d (b - c)^2.$$

L. Instead of deferring to the end of the operation, the investigation of the greatest common divisor, independent of the letter with reference to which the quantities are arranged, it is less troublesome to seek it at first, because, as it frequently happens, the remainders of each partial operation become with each new step more and more complicated, and the process more difficult as we advance.

Let there be, for example, the quantities,

$$a^4b^2 + a^3b^3 + b^4c^2 - a^4c^2 - a^3bc^2 - b^2c^4$$
$$a^2b + ab^2 + b^3 - a^2c - abc - b^2c.$$

After having arranged them with reference to the letter a, and uniting such terms as have the same factor, we have

$$(b^2 - c^2)\, a^4 + (b^3 - bc^2)\, a^3 + b^4c^2 - b^2c^4$$
$$(b - c)\, a^2 + (b^2 - bc)\, a + b^3 - b^2c$$

observing, in the first place, that if these quantities have a common divisor independent of a, it must divide each of the quantities multiplied by the different powers of a (49), as well as the quantities $b^4c^2 - b^2c^4$ and $b^3 - b^2c$, which do not contain this letter.

The question is reduced then to finding the divisors common to the two quantities $b^2 - c^2$ and $b - c$, and verifying afterwards whether amongst these divisors there is one that will divide at the same time

$$b^3 - bc^2 \text{ and } b^2 - bc,\ b^4c^2 - b^2c^4 \text{ and } b^3 - b^2c.$$

Dividing $b^2 - c^2$ by $b - c$, we find an exact quotient $b + c$; $b - c$ is therefore a common divisor of the two quantities $b^2 - c^2$ and $b - c$, which can evidently admit of no other, since the quantity $b - c$ is divisible only by itself and by unity. We must now ascertain whether $b - c$ will divide the other quantities referred to above, or whether it will divide in the same time the two proposed quantities; this is found to be the case; we have thus,

$$(b + c)\, a^4 + (b + bc)\, a^3 + b^3c^2 + b^2c^3.$$
$$a^2 + ba + b^2$$

In order to reduce these two last expressions to the greatest degree of simplicity, we must try if the first is not divisible by $b + c$; we find upon trial that it is. and there only remains to find the greatest common divisor of the very simple quantities

$$a^4 + ba^3 + b^2c^2$$
$$a^2 + ba + b^2.$$

Proceeding with these as the rule prescribes, we arrive, after the second division

$$\begin{array}{lrl}
a^2 + ba + b^2 & a^4 + ba^3 + b^2c^2 \qquad & a^2 - b^2 \\
 & -a^4 - ba^3 - a^2b^2 \qquad & \\ \hline
 & -a^2b^2 + b^2c^2 & \\
 & +a^2b^2 + ab^3 + b^4 & \\ \hline
\text{at the remainder} & + ab^3 + b^2c^2 + b^4 &
\end{array}$$

containing the letter a of the first power only; and as this remainder is not a common divisor, we conclude that the letter a does not make a part of the greatest common divisor sought, which therefore can be composed only of the factor $b - c$.

If, besides this common divisor, we had found another containing the quantity a, it would have been necessary to multiply together these two divisors in order to obtain the greatest common divisor sought.

After having acquired some practice in analysis, these remarks will enable the student to find in every case the greatest common divisor; we shall find without difficulty that the quantities

$$6a^5 + 15a^4b - 4a^3c^2 - 10a^2bc^2,$$
$$9a^3b - 27a^2bc - 6abc^2 + 18bc^3,$$

have for their greatest common divisor the quantity

$$3a^2 - 2c^2.$$

LI. The four *fundamental operations*, viz. addition, subtraction, multiplication and division are performed on algebraic fractions as in arithmetical fractions, observing only to proceed in the operations prescribed by the rules of arithmetic, according to the methods given for algebraic quantities. In referring here to those rules, we give an example of the application of each.

The sum of the fractions

$$\frac{a}{d}, \quad \frac{b}{d}, \quad \frac{c}{d}$$

which have the same denominator, or

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a+b+c}{d}$$

The difference of the fractions

$$\frac{a}{d} \text{ and } \frac{b}{d}$$

which have the same denominator is

$$\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}.$$

The whole number a being joined to the fraction $\frac{b}{c}$, the expression becomes

$$a + \frac{b}{c} = \frac{ac}{c}$$

Again the expression

$$a - \frac{b}{c} = \frac{ac}{c} - \qquad ac-$$

Reciprocally the expression

$$\frac{ac+b}{e} = \frac{ac}{c} + \frac{b}{c} = a + \frac{b}{c}$$

the expression $ac - b \quad ac \quad b$

LII. The fractions $\frac{c}{d}$ being brought to the same denominator become respectively

$$\frac{ad}{bd}, \frac{bc}{bd}$$

when the proposed fractions are equal, we find $ad = bc$; dividing the two numbers by cd and calling q the quotient, we have

$$\frac{ad}{cd} = \frac{a}{c} = q; \; \frac{bc}{cd} = \frac{b}{d} = q, \text{ whence } a \quad cq, \; b = dq$$

whence it appears that the two terms of one of the fractions are precisely these of the other multiplied by a common factor.

The fractions

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}$$

by the same reduction become respectively

$$\frac{adfh}{bdfh}, \frac{cbfh}{bdfh}, \frac{ebdh}{bdfh}, \frac{gbdf}{bdfh}.$$

In the note to pages 19, 20 and 21, we have given a method to arrive, in most cases, to a more simple denominator than that which results from the general rule. The algebraic symbols, facilitate in a great measure its application, as we shall see.

If, for example, we have the two fractions $\frac{a}{bc}, \frac{d}{bf}$, it is obvious, that if f were a factor of the 1st fraction and c of the second, the two denominators would be the same; therefore

multiplying the two terms of the first fraction by f, and the two terms of the second by c, we obtain the fractions $\frac{af}{bcf}$ and $\frac{cd}{bcf}$ which are more simple than $\frac{abf}{bbcf}$ and $\frac{bcd}{bbcf}$, which would have been obtained by the multiplication of each fraction by the denominators of all the other fractions.

It follows from this, that if the numerator and the denominator of a fraction contain the same letter, it can be cancelled without altering the value of the fraction.

In general, *to form the common denominator, we collect into one product all the different factors raised to the highest power found in the denominators of the proposed fractions; and it remains only to multiply the numerator of each fraction by the factors of this product, which are wanting in the denominator of the fraction.*

Having for example the fractions

$$\frac{\quad}{b^2c},\quad \frac{\quad}{bf},\quad \frac{\quad}{cg}$$

we form the product b^2cfg, and then multiply the numerator of the first fraction by fg, that of the second by bcg, and that of the third by b^2f, and we obtain:

$$\frac{afg}{b^2cfg},\quad \frac{bcdg}{b^2cfg},\quad \frac{b^2ef}{b^2cfg}.$$

LIII. For multiplication, we have

$$\frac{a}{b} \times c = \frac{ac}{b}, \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

For division,

be $\frac{a}{b}$ to be divided by c, then $\frac{a}{bc}$ or $\frac{a}{b} \times \frac{1}{c}$,

be $\frac{a}{b}$ to be divided by $\frac{c}{d}$ then $\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

The terms of the preceding fractions were simple quantities, but if we had fractions, the terms of which were polynomials, we should have to perform by the rules given for compound quantities, the operations indicated to be done upon simple quantities; thus we have;

$$\frac{a^2+b^2}{c+d} \times \frac{a-b}{c-d} = \frac{(a^2+b^2)(a-b)}{(c+d)(c-d)} = \frac{a^3+ab^2-a^2b-b^3}{c^2-d^2}$$

The quotient of the fraction,

$$\frac{a^2+b^2}{c+d} \text{ divided by } \frac{a-b}{c-d} \quad \frac{a^2+b^2}{c+d} \times \frac{c-d}{a-b}$$

$$= \frac{(a^2+b^2)\,(c-d)}{(c+d)\,(a-b)} = \frac{a^2c+b^2c-a^2d-b^2d}{ac+ad-bc-bd}$$

and so of other operations

LIV. Understanding what precedes, we can resolve any equation of the 1st degree, however complicated.

If we have, for example, the equation

$$\frac{(a+b)\,(x-c)}{a-b} + 4\,b = 2\,x - \frac{a\,c}{3\,a+b}$$

we should begin by making the denominators disappear, indicating only the operations; we shall then have

$(a+b)\,(x-c)\,(3\,a+b) + 4\,b\,(3\,a+b)\,(a-b) =$
$2\,x\,(a-b)\,(3\,a+b) - a\,c\,(a-b)$

performing then the multiplications indicated, we shall find

$3\,a^2\,x + 4\,a\,b\,x + b^2\,x - 3\,a^2c - 4\,a\,b\,c - b^2\,c + 12\,a^2\,b -$
$8\,a\,b^2 - 4\,b^3 = 6\,a^2\,x - 4\,a\,b\,x - 2\,b^2\,x - a^2\,c + a\,b\,c$

and transposing into one member, all the terms involving x, we shall obtain

$$-3\,a^2\,x + 8\,a\,b\,x + 3\,b^2\,x =$$

$2\,a^2\,c + 5\,a\,b\,c + b^2\,c - 12\,a^2\,b + 8\,a\,b^2 + 4\,b^3$, whence we conclude

$$x = \frac{2\,a^2\,c + 5\,a\,b\,c + b^2\,c - 12\,a^2\,b + 8\,a\,b^2 + 4\,b^3}{-3\,a^2 + 8\,a\,b + 3\,b^2}.$$

Examples in Division in which the Divisor is not an aliquot part of the Dividend

1. $\dfrac{1}{1-a} = 1 + a + a^2 + a^3 + a^4 +$ &c.

2. $\dfrac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 -$ &c.

$\dfrac{}{1+1} = 1 - 1 + 1 - 1 + 1 -$ &c.

4. $\dfrac{c}{a+b} = \dfrac{c}{a} - \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} - \dfrac{b^3c}{a^4} + \&c.$

5. $\dfrac{c}{a-b} = \dfrac{c}{a} + \dfrac{bc}{a^2} + \dfrac{b^2c}{a^3} + \dfrac{b^3c}{a^4} + \&c.$

6. $\dfrac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + 2x^4\ \&c.$

Examples in Reduction of Fractions.

7. $\dfrac{4}{7}x - \dfrac{3ab}{4d} + \dfrac{a}{2} - 1 = \dfrac{32dx - 42ab + 28ad - 56d}{56d}.$

8. $\dfrac{ab}{c} - \dfrac{ac}{d} + \dfrac{e}{b^2} - \dfrac{m}{2n} - b = \ldots\ldots\ldots\ldots\ldots\ldots$

$\dfrac{2ab^3dn - 2ab^2c^2n + 2cden - b^2cdm - 2b^3cdn}{2b^2cdn}.$

9. $\dfrac{3}{a} + \dfrac{1}{b} - \dfrac{1}{2d} + 4 = \dfrac{6bd + 2ad - ab + 8abd}{2abd}.$

10. $\dfrac{7a+3cd}{4} + \dfrac{9a-3cd}{4} - 3a = a.$

11. $5a - 5b - 10c + \dfrac{d}{2e} + \dfrac{m}{3n} = \ldots\ldots\ldots\ldots\ldots\ldots$

$\dfrac{30en\,(a - b - 2c) + 3dn + 2em.}{6en}$

12. $\dfrac{12a-7c}{5} - \dfrac{7a-7c}{5} - b = a - b.$

13. $a + 3bc - 5bd + \dfrac{abc - 3mn - 3b^2c^2}{bc - 2bd} = \ldots\ldots\ldots$

$2abc - 11\,b^2cd - 2abd + 10\,b^2d^2 - 3mn.$

14. $\dfrac{9a^2b - 3cd}{7} - \dfrac{13a^2b - 6\,cd}{14} + \dfrac{b - 6\frac{1}{2}a}{27} = \dfrac{135a^2b + 14b - 9}{378}$

15. $\dfrac{1-m}{a-x} - \dfrac{1-m}{a+x} = \ldots\ldots\ldots\ldots\ \dfrac{2x\,(1-m).}{a^2 - x^2}$

16. $\dfrac{(1+m)^2}{1-m^2} + \dfrac{(1-m)^2}{1+m^2} = \ldots\ldots\ldots\ldots\ \dfrac{2\,(1+m^2)}{1-m^4}.$

17. $\dfrac{a}{a+b} + \dfrac{b}{a-b} = \ldots\ldots\ldots\ldots\ldots\ldots\ \dfrac{a^2+b^2}{a^2-b^2}.$

18. $$\left(\frac{a}{a^2+b^2} - \frac{a}{a^2-b^2}\right)\left(\frac{a^4-b^4}{-2b}\right) = ab$$

19. $$\left[\frac{5a}{(a-2x)^2} + \frac{3a^2+x}{(a+x)(a-2x)^2}\right](2a+2x)$$

$$\frac{x(3a-1)}{a-2x^2}.$$

20. $$ac^2\left(x+\frac{5y^2}{c}\right) + (2b^2+5a)(c^2x-cy^2)$$

$$6c^2\left(ax - \frac{1}{3c}b^2y^2\right) - 2b^3c^2x = 0.$$

Simple Equations.

LV. Although no general precise rule can be given for forming the equation of any question whatever; there is notwithstanding a precept, the application of which, if well understood, will not fail to lead to the proposed end. This is it :

To indicate by the aid of algebraic signs, upon the known quantities, represented either by numbers or letters, and upon the unknown quantities, represented always by letters, the same reasonings and the same operations, which it would have been necessary to perform in order to verify the values of the unknown quantities, if these had been known.

In making use of this precept, we must in the first place determine with care, what are the operations which are contained in the enunciation of the question either directly or by implication ; but this is the very thing which constitutes the difficulty of *putting a question into an equation.*

We shall now give examples to illustrate the application of the above precept.

1. *There are two fountains, the first of which running for $2\frac{1}{4}$ hours, fills a certain cistern; the second fills the same cistern in $3\frac{3}{4}$ hours ; in what time will they fill it when both fountains run into the cistern in the same time?*

If the time were given, we should verify it by calculating the quantities of water discharged by each fountain during that time, and then adding the results, we should be certain that they compose the totality of water that the cistern can contain.

To form the equation we express the unknown time by x, and we indicate upon x the operations implied by the question ; but in order to render the solution independent of the given numbers, and in the same time to abridge the expressions of

those quantities where fractions are concerned, we will represent them also by letters; a being written instead of $2\frac{1}{2}$, and b instead of $3\frac{3}{4}$ h. This being supposed, by putting the capacity of the cistern equal to unity, it is evident, that the first fountain, which will fill it in a number of hours denoted by a, will discharge in one hour a quantity of water expressed by the fraction $\frac{1}{a}$, (for were the fountain to fill the cistern in 4 h. it would fill in one hour the $\frac{1}{4}$ part of it) and that consequently it would furnish in a number of x hours, the quantity $x \times \frac{1}{a}$ or $\frac{x}{a}$.

The second fountain, which will fill the same cistern in b hours, will discharge in one hour a quantity of water expressed by the fraction $\frac{1}{b}$; consequently in a number of x hours it will furnish the quantity $x \times \frac{1}{b}$ or $\frac{x}{b}$.

The total quantity of water then furnished by the two fountains will be

$$x \quad + \frac{x}{b};$$

But as this quantity of water must be equal to the quantity that the cistern is capable of containing which has been taken for unity, we have the equation

$$x \quad + \frac{x}{b} = 1.$$

This equation reduced by the preceding rules, becomes

$$b\,x + a\,x = ab$$

$$x = \frac{ab}{a+b}.$$

This last formula, gives this simple rule for resolving every case of the proposed question.

Divide the product of the numbers which denote the times employed by each fountain individually in filling the cistern by the sum of these numbers; the quotient expresses the time required by the two fountains running together to fill the cistern.

Applying this rule to the particular case given in enunciating the present problem, we have

$$2\tfrac{1}{2} \times 3\tfrac{3}{4} = \qquad \times \frac{15}{4} = \frac{75}{8}$$

$$2\tfrac{1}{2} + 3\tfrac{3}{4} = \frac{5}{2} + \quad 15 \qquad \frac{20}{8} \qquad \frac{30}{8} \qquad \frac{50}{8}$$

thence $x = \dfrac{75}{50} = \qquad$ or $1\frac{1}{2}$ hour.

2. *Let it be required to divide a number* a *into three parts having amongst themselves the proportions of the given numbers*, m, n, *and* p.

It is evident that the verification of the question would be as follows:

Expressing by x the 1st part, we shall get

$$m : n :: x : \text{the second part} \quad \frac{nx}{m}$$

$$m \quad p :: x : \text{the third part} = \frac{px}{m}$$

these three parts added together must make the number to be divided.

We have then the equation

$$x + \frac{nx}{m} + \frac{px}{m} = a$$

and by reducing all the terms to the same denominator m, the equation will be:

$$mx + nx + px = ma, \text{ or } x\,(m + n + p) = ma$$

and we deduce

$$x \qquad \frac{am}{m + n + p}$$

This result is only the algebraic translation of *the Rule of Fellowship* given in Arithmetic; for by regarding the numbers m, n and p as representing the stocks of several merchants trading in company, $m + n + p$ indicates the total capital, and a the gain to be divided, and the equation

$$x = \frac{am}{m + n + p}$$

indicates *that a share is obtained by multiplying the corresponding stock by the total gain, and dividing the product by the sum of the stocks;* which reduced to a proportion becomes *the total capital : a particular stock :: as the total gain : to the particular gain.*

LVI. The formation of the equation from the following problem requires an attention to some things which have not yet been considered.

3. *An archer, in order to encourage his son promises to give him 5 annas for every shot that should hit the dial, but requires that the son should forfeit 3 annas for every arrow that should pass beside it. After 40 shots the father and son settling their accounts, the former is found to owe the latter Rs. 8, 8 ans. How many of his shots hit the mark and how many missed it?*

If we represent the number of successful shots by x, the number of the unsuccessful ones will be, $40 - x$; if these numbers were given, we should verify them by multiplying the first by 5 annas, to ascertain how much the father has to pay the son, and the second by 3 annas, to obtain the number of annas which the son engaged to return to the father. The first sum must exceed the last by rupees $8\frac{1}{2}$ or 136 annas which the father owes the son.

The first sum then, is $5 \times x$
the second $3\ (40 - x)$.

The condition that the first sum must exceed the second by 136 annas, is expressed by the equation

$$5x - 3\ (40 - x) = 136 \text{ annas},$$
$$\text{or} \quad 5x - 120 + 3x = 136,$$
$$\text{or} \quad 8x = 256$$
$$\therefore \quad x = 32$$

the number of unsuccessful shots are consequently

$$40 - 32 = 8.$$

Indeed, $5 \times 32 - 3 \times 8 = 160 - 24 = 136$ ans. = Rs. $8\frac{1}{2}$.

To render the solution general, we may represent the sum which the father pays his son for every successful shot by a, and that which the son has to return the father, for every unsuccessful shot by b; by c the total number of shots, and lastly by d, the balance due by the father to the son, after the total number of shots. Representing, as before, by x, the number of successful shots, $c - x$ will be that of unsuccessful ones; every shot of the first kind being worth to the son the sum a, x shots would be worth $a \times x$ or ax, and the unsuccessful shots would be worth to the father the sum b; $c - x$ shots would be worth $b\ (c - x)$; which last sum subtracted from ax will give

$$ax - b\ (c - x) = d \text{ or}$$
$$ax - bc + bx = d$$

As this general formula indicates what operations must be performed on the given numbers a, b, c and d, in order to obtain the unknown quantity x, it may be translated into the form of a rule, or we may write, instead of the letters a, b, c, d, the given numbers; this last process is called *substituting* the values of the given quantities or putting the formula in numbers. Applying those of the foregoing problem, we obtain

$$x = \frac{136 + 3 \times 40}{5 + 3}$$

by performing the operations indicated we get, as before

$$x = \frac{136 + 120}{8} = \frac{256}{8} = 32.$$

4. *A town being seized with the plague,* $\frac{5}{8}$ *part of its inhabitants and* 120 *persons fell sick, of which* $\frac{1}{3}$ *part and* 145 *died; on which* 3 *times as many persons as had died and* 1100 *more, abandoned the town. After which a census being made, it was found, that the number of the inhabitants was reduced to* 340 *souls less than the* $\frac{1}{8}$ *th part, previous to the calamity. How many persons remained?*

Let x stand for the number of inhabitants, the number of sick is then expressed by

$$\frac{5}{8}x + 120$$

and those that died

$$\frac{1}{3}(\frac{5}{8}x + 120) + 145 \text{ or } \frac{1}{3}(\frac{5}{8}x + 555)$$

and the inhabitants leaving the place, is denoted by

$$\frac{5}{8}x + 555 + 1100, \text{ or } \frac{5}{8}x + 1655,$$

after which, the number of inhabitants is reduced to

$$\frac{x}{8} - 340$$

which quantity must also express, the number of inhabitants remaining in town, after deducting the dead and those that left the place, we have then the equation

$$x - \frac{1}{3}(\frac{5}{8}x + 555) - (\frac{5}{8}x + 1655) = \frac{x}{8} - 340$$

multiplying by 3 and then by 8, or at once by 24, and simplifying, the result of the equation becomes

$$x = 36000.$$

5. *A trader gives Rs.* 1200 *per annum for the expences of his family, after the first year he finds that he had increased his remaining fortune by one-fourth part; the 2nd year after defraying the same expences, he gained one-third of what he has remaining, and the 3rd year, he gained one-fifth of what remained after paying the usual domestic expences. He then found he had increased his primitive stock by* 50 *per cent. How much had he at first?*

Let his fortune be expressed by x,

then $x - 1200 + \frac{1}{4}(x - 1200)$ or $\frac{5}{4}x - 1500$

$=$ Remainder after the 1st year.

$\frac{5}{4}x - 1500 - 1200 + \frac{1}{3}(\frac{5}{4}x - 2700) = \frac{20}{12}x$

$- 3600 =$ Remainder after 2nd year.

$\frac{5}{3}x - 3600 - 1200 + \frac{1}{5}(\frac{5}{3}x - 4800) = \frac{6}{3}x$

$- 4800 - 960 = 2x - 5760 = 1\frac{1}{2}x.$

$$\therefore 5x = 57600 \text{ and } x = 11520$$

his first stock of trade.

6. *A cask capable of holding* 146 *gallons, was filled with a mixture, of wine, brandy and water. There were* 15 *gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?*

Let x represent the number of gallons of brandy,

$\therefore x + 15$ will represent that of wine,

and $2x + 15 =$ number of gallons of water,

$\therefore x + x + 15 + 2x + 15 = 146$

$\therefore \quad 4x + 30 = 146$ or $4x = 116$

$$x = 29$$

consequently there were 29 *gallons of brandy,* 44 *of wine, and* 73 *of water.*

7. *To remove four articles of furniture, I required for the* 1*st article two coolies, for the 2nd three, for the 3rd four and for the 4th five. After giving the first set of men one pile of pice and one pice more; to the 2nd set, one pile and four pice more; to the 3rd one pile and five pice more, and*

to the 4th one pile and nine pice more, I found that each man of the 3rd and 4th set had received the same number of pice. How many pice were in each pile; how many pice did each man receive, and how many pice did I distribute?

Let x represent the number of pice in one pile; then the 3rd set of men received, $x + 5$, and as four men were engaged, each received the fourth part $\frac{x+5}{4}$;

by the same reasoning, each man of the 4th set received, $\frac{x+9}{5}$

but since each cooly of the 3rd and 4th set received the same number of pice, we have

$$\frac{x+5}{4} = \frac{x+9}{5}$$

whence $x = 11$, the number of pice in a pile

each man of the 1st set received $\frac{x+1}{2} = 6$ pice;

ditto of the 2nd set received $\frac{x+4}{3} = 5$ pice;

ditto 3rd do. $\frac{x+5}{4} = 4$ pice

ditto 4th do. $\frac{x+9}{5} = 4$ pice

and I distributed $2 \times 6 + 3 \times 5 + 4 \times 4 + 5 \times 4 = 63$ pice.

3. *A Baboo, one-third part of whose property consisted of cash, and the remaining two-thirds of bad debts, bequeathed his estate to his four sons under the following conditions: that the eldest should come in for a quarter more than the share of the second son, who was to receive a third more than the share of the third son; to his fourth son, who was a merchant, he left the whole of his bad debts, of which by dint of perseverance, he recovered all but the $\frac{1}{7}$th part; had he got Rs.* 15,600 *more, he would have cleared a sum equal to three times the share of the second son. How much did each son receive?*

The operation will be facilitated by taking the property left to his third son as the unknown quantity, or x,

then the second son will have received $x + \frac{1}{3}x$ or $\frac{4}{3}x$

and the eldest $\frac{4}{3}x + \frac{1}{4} \times \frac{4}{3}x$ or $\frac{5}{3}x$

his whole property in cash was therefore the sum of these three shares $x + \frac{4}{3}x + \frac{5}{3}x = \frac{12}{3}x = 4x$.

The bad debts amounted to double that sum or $8x$, of which he recovered $8x \times \frac{3}{7}$ or $\frac{24}{7}x$

$\therefore \frac{24}{7}x + 15600 = 3 \times \frac{4}{3}x$, multiplying by 7

$$24x + 109200 = 28x$$

$\therefore x = 27300$, is the number of rupees the 3rd son received; consequently the 2nd received

$$27300 + \frac{27300}{3} = 36400.$$

The eldest received $36400 + \frac{36400}{4} = 45500$.

The cash amounted therefore to

$$27300 + 36400 + 45500 = 109200,$$

and the bad debts to twice that sum, or 218400;

and the share of the 4th son is the $\frac{3}{7}$th of that amount

$$\frac{3}{7}(218400) = 93600;$$

but had he received 15,600 more or 109,200, his share would have amounted to thrice the share of the second son, or $3 \times 36{,}400 = 109{,}200$; this solution agrees with the conditions of the question.

9. *A Gambler playing at cards, won four times as much money as he brought with him; he then lost Rs. 5, afterwards he had to pay for his expenses one rupee less than two-thirds of what he had remaining. On the next night, he brought only half the money he brought on the preceding night, and began by losing Rs. 7, 8 as. less than seven times what he had; after which he won by betting Rs. 225, 8 as. Paying then for his expenses, which amounted to half of what he had remaining: he found that he had as many rupees remaining as he brought home on the preceding night. How much did he bring with him on the first night?*

Supposing x was that number of rupees, then $x + 4x - 5$ or $5x - 5$ is the sum he had after gambling, of which he spent one rupee less than the $\frac{2}{3}$rd, viz. $\frac{2}{3}(5x - 5) - 1$

$$\therefore 5x - 5 - [\frac{2}{3}(5x - 5) - 1] = \frac{15x - 15 - 10x + 10 + 3}{3}$$

$$\frac{5x - 2}{3}$$

the sum he brought home on the first night.

As gains are treated as positive quantities, losses must be considered as negative quantities, therefore on the second night, when he brought with him only $\frac{x}{2}$, his losses and gains should be expressed

$-(7\times\frac{1}{2}x-7,8\ as.)+225,8\ as.$ which added to $\frac{x}{2}$, the sum he brought with him, is $\frac{x}{2}-\frac{7}{2}x+7,8\ as.+225,8\ as.=$ the sum remaining, and paying one-half of it, he had the other half equal to

$\dfrac{\frac{x}{2}-\frac{7}{2}x+7,8\ as.+225,8\ as.}{2}=\dfrac{-6x+466}{4}$; but this remainder is the same as that of the preceding night, viz. $\frac{5x-2}{3}$, we have then the equation

$$\frac{5x-2}{3}=\frac{466-6x}{4}$$

whence $x=37$.

10. *A woman was carrying eggs to market, to defray the expences on her way thither, she gave three eggs less than ⅓rd part of the total number of her eggs. On her return from market, where she had sold the remaining of her eggs at the same price of one pice a piece, she spent ⅕th part of the proceeds for her expences home, which amounted only to the half of her preceding expences. How many eggs had she on setting out?*

If x be the number of eggs she had, her expences on her way to the market, are then denoted by

$$\frac{x}{3}-3$$

Subtracting this quantity from x, we get a remainder

$$\frac{2x}{3}+3$$

the ⅕th part of which, being the half of what she had spent at first, we obtain the equation,

$$\therefore \frac{1}{2}\left(\frac{x}{3}-3\right)=\frac{1}{5}\left(\frac{2x}{3}+3\right)$$

$$\therefore \text{ or } 5x-45=4x+18$$

$$\therefore x=63.$$

11. *A gentleman unacquainted with numbers, bequeaths his estate, valued at Rs.* 62,250, *to his three friends A, B and C ; to A he bequeathed the half of his property plus Rs.* 1,000, *to B the* $\frac{1}{3}$ *part plus Rs.* 2,000, *and to C, the* $\frac{1}{4}$ *plus Rs.* 4,000. *How much was each to receive agreeably to the testator's will?*

It is evident that the sum of the three shares must make up the whole property, and as the sum of the three fractions alone surpasses unity, we must look for a number, of which the three shares, agreeably to the testator's will, amount to Rs. 62,250.

Be that number x

then $\frac{x}{2} + 1000$ will be A's share,

$\frac{x}{3} + 2000$ .... B's share,

and $\frac{x}{4} + 4000$ .... C's share,

$\therefore\ 6x + 4x + 3x + 7000 \times 12 = 62250 \times 12$

or $13x = 663000 \qquad \therefore\ x = 51000$

Substituting this value of x in the 3 shares of *A*, *B* and *C*

A's share $= 25500 + 1000 = 26500$
B's share $= 17000 + 2000 = 19000$
C's share $= 12750 + 4000 = 16750$

Sum of the three shares or the amount bequeathed $= 62250$.

12. *A man when he married was three times as old as his wife ; but* 18 *years after, he was only twice as old ; what were their ages on their wedding day?*

Suppose the woman's age was x years,
then that of the man was $3x$
but 18 years after, each was 18 years older, and then his age was double that of the woman's, which, translated into algebraic language, is indicated by the equation,

$\therefore\ 2(x + 18) = 3x + 18$ or

$2x + 36 = 3x + 18 \quad \therefore\ x = 18$, the woman's age, and $3 \times 18 = 54$ years, was the man's age.

13. *When a boy was asked how many geese he had, he replied : If I had as many more, plus a third part as many more, and nine and one-third geese, I should have* 308 *geese. How many geese had he?*

Suppose he had x geese ; then by the nature of the question

$x + x + \frac{x}{3} + 9\frac{1}{3} = 308$, multiplying by 3

$3x + 3x + x + 28 = 924$ or

$7x = 896 \qquad \therefore\ x = 128.$

14. *A Baboo bought a horse at an auction, but finding it to be unsound, he sent it to a Veterinary Surgeon, to whom, as it was only apparently cured, he agreed to pay* 12 *per cent. of the clear produce of a resale by Auction, viz: after deducting the Auctioneer's commission of* 8 *per cent. which amounted to Rs.* 38. *The Baboo then found he had lost* 20 *per cent. on his bargain. How much did he pay for the horse?*

Let x represent the number of rupees the horse was sold for on the Baboo's account, of which 8 Rupees in a hundred, amounted to Rs. 38: we have then the equation

$$\frac{8\,x}{100} = 38$$

$$\text{or } 8\,x = 3800 \text{ and } x = 475 \text{ the resale}$$

the clear produce of which was $475 - 38 =$ Rs 437 of which sum the veterinary's charges of 12 per cent. must be deducted, viz. $437 - \frac{437 \times 12}{100}$ or $437 - 52,44 =$ Rs. 384,56, the sum the Baboo received after deducting all charges, but this is 20 per cent. less than what he paid for the horse; supposing he paid y Rupees for it, we may then form the equation:

$$y - \frac{20\,y}{100} = 384,\ 56$$

$$\text{or} \quad 100\,y - 20\,y = 38456$$

$$\therefore y = Rs.\ 480,\ 7$$

the price he paid for the horse.

15. *A sailor in a tavern borrowed half as much money as he had about him and then spent Rs.* 3 : 8 *ans. Going to a second tavern he borrowed twice as much money as he had remaining and then spent Rs.* 1 : 8 *ans. From thence, he went to a third tavern, and after borrowing as much money as he then had and paying for his expences, which amounted to Rs.* 2 : 8, *he proceeded to a fourth tavern, but finding no one there who would lend him money, he spent only his last eight anas remaining. How much money had he at first? how much did he borrow, and what did he spend?*

Supposing he had x Rupees, then in 1st tavern, $x + \frac{x}{2} - 3:8 = \frac{3}{2}x - 3:\ 8$ is what he had remaining.

$\frac{\ }{2}$ [illegible] $9\,x$

$-$ 12, the remainder on leaving the second tavern.

In 3rd tavern, $\frac{9}{2}x - 12 + \frac{9}{2}x - 12 - 2:8 = 9x - 26:8$ is what he had remaining.

In 4th ditto, we get the equation $9x - 26 : 8 = 8$ annas,

$$\text{or } 9x = 27 \quad \therefore x = 3.$$

The sum he borrowed was then as follows: in the first tavern $\frac{x}{2}$; in the 2nd, $3x - 7$; and in 3rd, $\frac{9}{2}x - 12$, and substituting Rs. 3 for x, we get

$1:8 + 9 - 7 + 13:8 - 12 =$ Rs. 5, the sum he borrowed, and having nothing left on leaving the 4th tavern, the sum Rs. 5 which he borrowed, plus the sum he brought with him, Rs. 3, make together Rs. 8, which ought to be equal to the sums he spent in the four taverns, viz.

$$3:8 + 1:8 + 2:8 + 0:8 = \text{Rs. } 8.$$

16. *Eighteen pupils of the* 1*st and* 2*nd Classes obtained prizes; the value of a prize of each scholar of the* 1*st Class is Rs.* 4 : 8, *and that of a prize of each scholar of the* 2*nd Class Rs.* 3 : 8 : *the value of all the* 18 *prizes is Rs.* 70. *How many pupils were there of each class?*

Let the number of pupils of the 1st Class be x.
Then the number of pupils of the 2nd Class will be expressed by $18 - x$, we then have the equation.

$$4:8 \times x + 3:8\,(18 - x) = 70$$

$$\text{or } 4:8 \times x + 63 - 3:8 \times x = 70$$

$$x = 7$$

the number of pupils in 1st Class;
then $18 - 7$, or 11, is the number in the 2nd class.

17. *A boy being asked how many geese he had, replied: if I had as many more and one third as many more and* $9\frac{1}{3}$ *geese, I then should have* 308 *geese. How many had he?*

Supposing he had x geese, then we get the equation

$$x + x + \frac{1}{3} + 9\frac{1}{3} = 308$$

$$7x = 924 - 28$$

$$\therefore x = 128.$$

Equations of the First degree with one unknown quantity.

To find the value of x.

1. $5x - 7 + 19 = 37$ .... Ans. $x = 5$.

2. $10 [illegible] 3x + 12 + x = \frac{x}{2} - 40 + 3x + 86$ Ans. $x = 48$.

3. $\frac{x}{3} + \frac{8}{3} + 2x = 9 - 5x + 21 + 10$ Ans. $x = 5$.

4. $\frac{8x - 11}{7} + \frac{7x}{2} - \frac{x}{7} \quad \frac{11x - 13}{2}$ Ans. $x = 4\frac{13}{14}$.

5. $\frac{6 - x}{4} + \frac{8 - 2x}{3} + \frac{10 - 3x}{2} \quad \frac{- 10}{4} + \frac{2x - 8}{3}$

$\frac{x - 6}{2}$ Ans. $x = 4$.

6. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} = \frac{7}{10}$ Ans. $x = 1\frac{1}{5}$

7. $\frac{x + 3}{2} + \frac{x}{3} = 4 - \frac{x - 5}{}$. Ans. $x = 3\frac{6}{13}$,

8. $\sqrt{12 + x} = \sqrt{x} + 6$ ... Ans. $x = 4$.

9. $x - a = \frac{x^2}{a^2 + a}$ .... Ans. $x = \frac{a^2}{a - 1}$

10. $\sqrt{x} + \sqrt{a + x} = \frac{2a}{\sqrt{a + x}}$ ...... Ans. $x = \frac{a}{3}$

11. $\frac{ax - b}{3} + \frac{ax}{4} \quad b^2 x \quad \frac{bx - ax}{4} + \frac{d}{12}$

Ans. $\frac{[illegible]\, d}{4[illegible] - 4b^2 + 3b}$.

12. $\sqrt{a + x^2} = \sqrt[4]{b^4 + x^4}$ .. Ans. $x = \sqrt{\frac{b^4 - a^4}{2a^2}}$.

13. $\sqrt{a + x} + \sqrt{a - x} = \sqrt{xa}$ Ans. $x = \frac{4a^2}{a^2 + 4}$.

14. $\frac{a}{1 + x} + \frac{a}{1 - x} + = b$ ... Ans. $x = \sqrt{\frac{b - 2a}{b}}$.

15. $a + x = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$ Ans. $x = \frac{b^2}{4a} - a.$

16. $\frac{1}{2}\sqrt{x^2 + 3a^2} - \frac{1}{2}\sqrt{x^2 - 3a^2} = x\sqrt{a}$

Ans. $x = \sqrt[4]{\frac{9a^2}{4 - 4a}}.$

17. $\frac{\sqrt{a + x}}{3} + \frac{\sqrt{a - x}}{3} = b$ Ans. $x = \frac{3b}{2}\sqrt{4a - 9b^2}.$

18. $\sqrt[3]{a + x} + \sqrt[3]{a - x} = b;$

Ans. $x = \sqrt{a^2 - \left(\frac{b^3 - 2a}{3b}\right)^3}.$

19. $\sqrt{a} + \sqrt{x} = \sqrt{ax}$ ...... Ans. $x = \frac{a}{(\sqrt{a} - 1)^2}$

20. $\sqrt{a^2 - ax} = a - \sqrt{a - ax}$ Ans. $x = \frac{4a - 1}{4a}.$

21. $\sqrt{x + a} = c - \sqrt{x + b}$

Ans. $x = \left(\frac{c^2 + b - a}{2c}\right)^2 - b.$

22. $\sqrt{\frac{b}{a + x}} + \sqrt{\frac{c}{a - x}} = \sqrt[4]{\frac{4bc}{a^2 - x^2}}$

Ans. $x = \frac{a(b + c)}{b - c}$

23. $x + \sqrt{a^2 + x^2} = \frac{2a^2}{\sqrt{(a^2 + x^2)}}$ Ans. $x = \frac{a}{\sqrt{3}}.$

24. $\frac{cx^n}{a + bx} = \frac{ax^n}{d + cx}$ .... Ans. $x = \frac{a^2 - dc}{c^2 - ab}$

25. $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \frac{3}{2}\sqrt{\frac{x}{x + \sqrt{x}}}$

Ans. $x = \frac{25}{16}.$

26. $\frac{1}{x} + \frac{1}{a} = \sqrt{\frac{1}{a^2} + \sqrt{\frac{4}{a^2x^2} + \frac{9}{x^4}}}$ Ans. $x = 2a.$

27. $\sqrt{a^2 + x^2} = \frac{\sqrt{b^4 + x^4}}{\sqrt{a^2 - x^2}}$ ... Ans. $x = \sqrt[4]{\frac{a^4 - b^4}{2}}.$

28. $\sqrt{27+x}+\sqrt{x}=\dfrac{45}{\sqrt{27-\dfrac{2x}{9}}}$ Ans. $x=9$.

29. $\dfrac{\sqrt{x}+2x}{\sqrt{x}+b}=\dfrac{\sqrt{x}+4a}{\sqrt{x}+3b}$ ...... Ans. $x=\left(\dfrac{ab}{a-b}\right)^2$

30. $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$ .... Ans. $x=\dfrac{1}{1-a}$

31. $\dfrac{ax-b^2}{\sqrt{ax}+b}=c+\dfrac{\sqrt{ax}-b}{c}$

Ans. $x=\dfrac{1}{a}\left(b+\dfrac{c^2}{c-1}\right)^2$

32. $\dfrac{3x-1}{\sqrt{3x}+1}=1+\dfrac{\sqrt{3x}-1}{2}$ .. Ans. $x=3$.

33. $\left\{\begin{array}{l}\dfrac{\sqrt[3]{x}+\sqrt{x}+x}{19}=\sqrt[3]{x}\\ 7\sqrt[3]{x}=x-4\frac{1}{2}\sqrt{x}\end{array}\right\}$ ... Ans. $x=64$.

18. *I have* 80 *seers of coffee, of a quality I sell at* 13 *anas the seer, and a large quantity of an inferior sort, which I sell at* 8 *anas per seer; how many seers of this last quality must I mix with the first to sell it at* 9 *anas the seer?*

Ans. 320.

19. *Strewing a rectangular place, whose sides are in the proportion of* 7 : 5, *with pebbles, it stood me at the rate of* 15 *anas the cubit, and then enclosing it with an iron railing at Rs.* 2 *per cubit, cost me three times as much. What was the length and breadth of the place?*

Ans. $3\frac{31}{75}$ *and* $2\frac{23}{51}$.

20. *The cost of paving with marble a path of* 4 *feet breadth round my rectangular garden, which is* 8 *feet longer than broad, amounted to Rs* 2,800, *at the rate of R.* 1 : 12 *the square foot. How long is the garden?*

Ans. 108 *feet.*

21. *A person has a certain number of rupees in his right hand, and another number in his left hand, in taking out one rupee from his right hand and putting it in his left, he finds the number of rupees in his right hand equal to that in his left; but if he takes out one from his left hand and puts it in his

right, he finds the sum in his right hand just double that in his left. Required the number of rupees he had in each hand?

Ans. 5 *and* 7.

22. *A person went to a market with* 50 *rupees and bought* 50 *yards of muslin and jean together, for muslin he paid* 2 *rupees a yard, and for each yard of jean he paid* 8 *anas. Required the number of yards of each sort?*

Ans. $16\frac{2}{3}$ *and* $33\frac{1}{3}$.

23. *There are three numbers in arithmetical progression whose sum is equal to* 21; *and the product of the sum of the first and third by the common difference, is equal to four times the second term. Required the numbers?*

Ans. 5, 7, *and* 9.

24. *Four brothers* A, B, C *and* D, *occupy a piece of ground measuring* $12\frac{1}{4}$ *biggahs: the portion which* A *occupies is worth* 1500 *rupees per biggah, that of* B, 900 *rupees, that of* C, 600 *rupees, and that of* D, 500 *rupees; but the portion of ground of each being of the same value, it is required to ascertain the number of biggahs each of the four brothers possesses.*

A's *share* $1\frac{1}{2}$
B ,, $2\frac{1}{2}$
C ,, $3\frac{3}{4}$
and D ,, $4\frac{1}{2}$.

25. *Find out two numbers, such that if* 8 *be added to the second, the sum will be twice the first number, but if* 8 *be subtracted from the first, the difference will be one-fourth of the second?*

Ans. 12 *and* 16.

26. *There are two numbers in the proportion of two to three and the sum of their squares is equal to* 468. *Required the numbers?*

Ans. 18 *and* 12.

27. *On planting a certain number of trees in the form of a square I had* [illegible]*er, but increasing every row by* 5, *I wanted* 949 *to complete the square. Required the number of trees?*

Ans. 10,500.

28. A *was caught by* B *stealing apples, who insisted on having half of what he had and six apples more, the same treatment he met with from* C, D, *and* E, *the last leaving him only* 12 *apples. How many had he at first?*

Ans. 372.

29. A, B, *and* C, *try their strength by throwing the discus, their throws added together were* 324 *feet;* A, *threw a certain number of feet,* B *as many as* A *and* 15 *over, but* C, *missing*

his throw, reached only a fifth part of their throws added together. What distance did each throw?

A $127\frac{1}{2}$, B $142\frac{1}{2}$, *and* C 54.

30. *A woman went to market with fowls and ducks' eggs, of which the number of the last exceeded that of the first by* 210. *She gave two of the first for three pice, and sold the whole of her eggs of both sorts for Rs.* 5 : 3 *ans.* 9 *p.; but* 20 *eggs of the first sort bring her in* 10 *pice more than* 30 *of the latter sort. How many fowls and how many ducks' eggs had she, and at what rate were the latter sold?*

Ans. 90 *and* 300, 3 *ducks' eggs for* 2 *pice.*

31. *Of a fishing rod consisting of two parts, the upper part is to the lower as* 5 *to* 7, *and* 9 *times the upper part together with* 3 *times the lower, exceeds the whole length of the rod by* 86 *inches. Required the length of the rod.*

Ans. upper 45, *lower* 63.

32. *A steamer to go down the river from* A *to* B, *a distance of* c *miles, through the middle of the stream, takes* a *hours; in returning it moves nearer the bank, where the stream loses* $\frac{1}{3}$ *of its strength, it requires* b *hours. What is the velocity of the stream per hour, and at what rate would the steamer ply in smooth water?*

Ans. $\dfrac{3c(b-a)}{5ab}$ = *velocity of the stream, and* $\dfrac{c(3a+2b)}{5ab}$ =*rate per hour, of the steamer.*

33. *To divide a number into* 4 *shares, so that the first share is increased by* 3, *the second diminished by* 3, *the third multiplied by* 3, *and the fourth divided by* 3, *the four shares shall be equal. The sum of the four shares being equal to seven times the* 1*st share* ~~*together with*~~ diminished by *the* 3*rd share increased by* $\frac{1}{3}$.

Ans. 13, 19, $5\frac{1}{3}$, 48.

34. *I bought four empty casks, and found on trial, that to fill the first cask from the second, required* $1\frac{5}{8}$ *the contents of the second, and filling the second cask from the third, it leaves the third one-twenty-fifth full, but filling the third cask from the fourth, leaves it one-fifth full; lastly, emptying the* 4*th cask into the* 1*st, there wanted* 51 *gallons to fill it. How many gallons does each cask hold?*

Ans. 176, 96, 100, 125.

35. *A quantity of rupees were to be distributed as prizes to six boys, the* 1*st prize was half the whole sum and half a rupee more, the* 2*nd was half the remainder and half a rupee more, and so on for the* 3*rd,* 4*th and* 5*th prizes; the* 6*th was*

one rupee which was all that was left. How many rupees were distributed, each prize consisting of a whole number of rupees?

Ans. 63.

36. *The ingredients of a loaf of bread weighing* 16 *lbs. were flour, salt and water. Adding* [illegible] *lbs. to the quantity of salt will make it equal to the weight of the flour; and the weight of the water together with that of the salt is the 3rd part of the weight of the flour. What were the weights of the three ingredients?*

Flour 12 *lbs., water* 3 *lbs., salt* 1 *lb.*

37. *A packet sailing from Dover with a fair wind, arrives at Calais in two hours, and on its return, the wind being contrary, it proceeds six miles an hour slower than it went; now when it is half way over, the wind changing, it sails two miles an hour faster, and reaches Dover sooner than it would have done had the wind not changed, in the proportion of* 6 : 7. *Required the rates of sailing, and the distance between Dover and Calais.*

Ans. The distance is 22 *miles, and in returning it sails* 5 *and* 7 *miles an hour.*

38. *Six hundred persons voted upon a disputed question, which was lost by a certain number. The same number of persons having voted again upon the same question, it was, from some change in circumstances, carried by twice as many as it was before lost. How many changed their minds?*

Ans. 150.

39. *The gas contractors engage to light a shop with* 5 *large and* 3 *small burners, but having by them only one large burner, supply the deficiency with five small ones. The shopkeeper not finding this light sufficient, procures two more small burners, and at the same time agrees for the light to burn double the usual time on Saturday nights, for which additional gas he was required to pay* £1. 11*s. How much did he pay altogether.*

Ans. 5 *guineas.*

40. A *and* B *set out from two places,* C *and* D, *at the same time towards* E; *the road from* C *to* E *being through* D. A *travels* 7 *miles an hour, and at that rate of travelling would have overtaken* B 5 *miles before he got to* E, *but after arriving at* D, *he travels* $6\frac{2}{3}$ *miles an hour, in consequence of which he overtakes* B *just as he enters* E. *Supposing* B *to travel* 5 *miles an hour, what are the distances between* C, D, *and* E.

Ans. From C *to* D 14 *miles, and from* D *to* E 40.

41. *A gentleman wishing his two daughters to receive equal portions when they became of age, bequeathed to the elder the accumulated interest of a certain sum of money, bought at the time of his death into the* 4 *per cent. stock at* 88 ; *and to the younger the accumulated interest of a sum less than the former by* £3,500, *bought at the same time into the* 3 *per cent. at* 63. *Supposing their ages at the time of their father's death to have been* 17, *and* 14, *what would be the sum bought into the stocks in each case, and what would be the fortune of each.*

Ans. The sums would be £7,700, *and* £4,200 ; *and fortune* £1,400.

42. *Two persons* A *and* B, *start at the same time for a race which lasted six minutes. Now after galloping four minutes at the same uniform pace at which each started, the distance between them is* $\frac{1}{440}$*th part of the whole length of the course. They continue to run for one minute more at the same speed as at first ; and then* B, *who is last, quickens the speed of his horse* 20 *yards a minute, and comes in exactly two yards before* A, *whose horse had run at the same uniform pace throughout. It is required to find the length of the course?*

Ans. 3 *miles.*

43. *Out of a common pack of cards, a certain number including the* 10 *of diamonds, was dealt equally amongst four persons, the dealer turning up the last card, which was the ten of spades, which he gave himself. Now if twice the number of cards had been dealt to each, the ten of spades being turned up by the dealer, and the ten of diamonds being still dealt out, the chance of the dealer's having the ten of diamonds would be to the chance against him as* 3 *to* 10. *Required the number of cards dealt to each the second time.*

Ans. 10.

44. *Two companies of soldiers consisting of equal numbers were sent out under* A *and* B, *from two hostile camps, to reconnoitre. Falling in with each other, a skirmish ensued in which* A *lost* 50, *killed and prisoners, and* B *had* 20 *killed.* A *however having been reinforced by a party equal to five-sevenths of the number which* B *had remaining, and* B *having been reinforced by a number greater by* 46 *than three-fifths of the number which* A *had remaining, they renewed the engagement, when* A *was forced to retire with the additional loss of* 30 *men. When the returns were made,* B *found he had again lost* 20 *men, but that he had then twice as many men remaining as* A *had. How many had each at first?*

Ans. 90.

45. *A sportsman, who kept an account of the number of birds which he killed, found that each succeeding season he wanted* 50, *in order that the number killed might bear the proportion of* 3 : 2 *to the number killed in the preceding year. In the fourth year he found that he had killed* 170 *fewer than three times the number killed in the first year. How many did he kill the first year?*

Ans. 180.

46. *Several detachments of Artillery divided a certain number of cannon balls. The first took* 72, *and one-ninth of the remainder; the next* 144, *and one-ninth of the remainder; the third* 216 *and one-ninth of the remainder; the fourth* 288 *and one-ninth of those that were left, and so on, when it was found that the balls had been equally divided. Determine the number of detachments and balls.*

Ans. 4608 *balls, and* 8 *detachments.*

47. A *entered into a canal speculation with* 14 *others, and the profits of this concern amounted in all to* £595 *more than five times the price of an original share, seven of his former partners in this affair, joined with him in a scheme for navigating the said canals with steam-boats, each venturing a sum of money less than his former gains by* £173. *But the steam-boats unexpectedly blowing up,* A *found he had lost* £419 *by them, for the company not only never recovered the money advanced, but had lost all they had gained by digging the canals and* £368 *besides. What were the prices of shares in the two concerns originally?*

Ans. £700 *in the first speculation, and* £100 *in the second.*

48. *A merchant wishing to buy a certain quantity of pimenta, the price of which he calculates at the rate of* 5 *bags for* £8, *transmits to his foreign agent the requisite sum of money. Before the order arrives, pimenta has risen in value; and the money is sufficient only to buy a quantity less by* 18 *bags than that which the merchant intended. It appears also that as many bags as exceed one-third of the intended quantity by* $5\frac{1}{4}$, *will now cost* £10 : 7*s. more than they would have done, had the price not varied. What is the quantity purchased?*

Ans. 432 *bags.*

49. *Four men walking abroad found a purse containing shillings only, out of which every one of them took a number at a venture. Afterwards comparing their numbers together, they found that if the first took* 25 *shillings from the second, it would make his number equal to what the second had left. If the second took* 30 *shillings from the third, his money would then be triple what the third had left. And if the third took* 40 *shillings from the fourth, his money would then be double*

what the fourth had left. Lastly, the fourth taking 50 *shillings from the first, he would then have three times as much as the first had left, and five shillings over. What had each.*

Ans. 100, 150, 90, *and* 105 *shillings respectively.*

50. *Fifteen current guineas should weigh* 4 *ounces; but a parcel of light gold being weighed and counted, was found to contain* 9 *more guineas than was supposed from the weight; and a part of the whole, exceeding the half by* 10 *guineas and a half, was found to be* $1\frac{1}{2}$*oz. deficient in weight. What was the number of guineas?*

Ans. 189.

51. *A merchant bought a quantity of wheat for* £200, *half of which he reserved for his private use. He then sold* 5 *bushels more than* $\frac{3}{4}$ *of the remaining quantity at such a price as to gain* 40 *per cent. But the price of wheat having advanced, he sold the remainder at such a price as to gain* 67 *per cent. by what he sold. And had the whole been sold at this latter price he would have gained* 160 *per cent. How much did he buy, and how did he sell it?*

Ans. He bought 400 *bushels; and sold the first portion at* 14 *shillings, and the second at* 26 *shillings per bushel.*

52. *Four jolly companions* A, B, C *and* D *began to drink a hogshead of beer;* A, B *and* C, *used to finish a hogshead in* $19\frac{63}{107}$ *days;* A, B *and* D *emptied the same quantity in* $17\frac{3}{111}$ *days;* B, C *and* D *took* $16\frac{44}{391}$ *days; and to* A, C *and* D *a hogshead lasted only* $15\frac{15}{29}$ *days. In what time will each alone finish a hogshead, and how many days will the same quantity last, if all four drink in company?*

Ans. A *in* 60 *days,* B *in* 70, C *in* 50, *and* D *in* 36

$$A + B + C + D \text{ in } 12\frac{87}{124} \text{ days.}$$

53. *A boy took* a *minutes to eat up an apple pie by himself alone, another finished a similar pie in* b *minutes, a third in* c *minutes, and a fourth in* d *minutes. Happening to have only one pie amongst them four, each eat with double avidity. In what time will they finish it?*

Ans. $\dfrac{abcd}{2\,(abc + abd + acd + bcd)}.$

54. *Engaging a number of coolies to remove my furniture, I found that by giving each* b *pice, I should expend* c *pice less than I assigned for that purpose, but that I should be losing* d *pice in giving each cooly* a *pice. How many coolies were employed; and how much did each get?*

Ans. *The number of coolies employed, was......* $\frac{c+d}{a-b}$

The number of pice to each cooly, .. $\frac{ac+bd}{a-b}$.

55. *Which is that number, whose product by* 7 *is as much above* 30, *as the quotient of that number by* 7, *is under* 30?

Ans. $8\frac{2}{5}$.

56. *In order to meet my expences, I ought to have a monthly income of Rs.* 715—*but it wants considerably of that sum. Were my income* $4\frac{1}{2}$ *times as great as it really is, then I should be able, not only to meet my expenses, but even to save as much above my income, as I am now deficient. What was my income?*

Ans. Rs. 260.

57. *A wine merchant has* 80 *cases of gin, each containing* 15 *square bottles, which he sells for Rs.* 18. *Since, however, he deems this price too high for his customers, he wishes, as he intends to deal fairly, to add as much water to it, as will enable him to sell the case of mixed gin for Rs.* 11. *How many square bottles of water must he add to it?*

Ans. $763\frac{3}{13}$ *square bottles.*

58. *My purse contains* 471 *pieces of Sicca and Company's Rupees, the joint value amounts to* 491 *Company's Rupees. How many of each sort are there in it?*

Ans. 300 *Sicca and* 171 *Company's Rupees.*

LVI. The questions which we have hitherto considered, involve only one unknown quantity, by means of which, with the known quantities, all the conditions of the question have been expressed. It is often found more convenient, in some questions, to employ two unknown quantities; but then there must be, either expressed or implied, two conditions, in order to form two equations, without which the two unknown quantities cannot be determined independently of each other, viz. the values of the unknown quantities cannot be expressed in known quantities alone.

The question in art. 5, especially as it is enunciated in art. 7, presents itself more naturally when expressed with two unknown quantities, viz. with both the numbers sought.

Indeed if we denote

the lesser number by	x
the greater by	y
their sum by	a
their difference by	b

we shall have agreeably to the enunciation of the question

$$y + x = a$$
$$y - x = b$$

Each of these two equations being considered by itself, we cannot determine one of the unknown quantities. If, for example, we deduce the value of x from the 1st equation, we get

$$x = a - y$$

a value which seems at first to teach us nothing with regard to what we are seeking, since it contains the quantity y which is unknown as well as x; but if, in lieu of the unknown quantity x, in the second equation, we substitute this value, the second equation containing now only one single unknown quantity y, will give the value of y by the same way as in the preceding examples.

Indeed, by this substitution we have

$$y - (a - y) = b$$

or $$2y - a = b$$

or $$y = \frac{a + b}{2}$$

and substituting this value of y in the first equation, then

$$\frac{a + b}{2} + x$$

or $$2x + a + b = 2a$$

$$x = \frac{a - b}{2}.$$

which are the same expressions for the unknown quantities as in art. (7).

It is easy to see, that the above solution does not differ in reality from that of (7) only we have supposed and resolved, the second equation $y - x = b$, which we contented ourselves with enunciating in common language in the article cited, and concluding, without any algebraic calculation, that the greater number was $x + b$.

Let us take other questions.

59. *A journeyman working at a baboo's house,* 12 *days, and having with him during the first* 7 *days, his two sons, received Rs.* 4 : 10; *being afterwards employed for eight days on the same conditions, and having his sons with him during five days, he received only Rs.* 3 : 2 *for this time. It is required to know how much he earned per day for himself alone, and how much in the same time did his two sons earn?*

Let x be the daily wages of the man,
y, that of his two sons;
12 days' work of the man will amount to $12\,x$,
7 days' ditto, his two sons, $7\,y$;
then by the first conditions of the question

$$12\,x + 7\,y = \text{Rs. } 4 : 10 = 74 \text{ anas.}$$

Again, 8 days' work of the man will give $8\,x$,
5 ditto, of his two sons, $5\,y$;
then by the second conditions of the problem

$$8\,x + 5\,y = \text{Rs. } 3 \text{ ,, } 2 = 50 \text{ anas}$$

and by a similar reasoning as in the preceding question, we take the value of y from the 1st equation, which is

$$y \quad 74 - 12$$

substituting this value of y, in the second equation, we get

$$8\,x + 5\,\frac{(74 - 12\,x)}{7} = 50,$$

which contains but one unknown quantity x.

By reducing it we have

$$56\,x + 370 - 60\,x = 350$$

or

$$370 - 4\,x = 350,$$

and transposing $-\,4\,x$ into the 2nd member, and 350 to the first, we get

$$370 - 350 = 4\,x$$
$$20 = 4\,x$$

or

$$5 = x.$$

Knowing the value of x, which we have just found equal to 5, we substitue this value in the formula

$$y = \frac{74 - 12\,x}{7},$$

the second member will be determined, for we have

$$74 - 12 \times 5 \quad 74 - 60 \quad \frac{14}{7} = 2$$

$$\therefore y = 2.$$

The man then earned 5 anas a day, whilst the two children together only 2 anas.

LVII. The reader must have observed, that in resolving the above equation $370 - 4x = 350$, we have transposed $4x$ to the second member: we have done so in order to avoid a slight difficulty, that would otherwise have occurred, and which we will now explain.

By leaving $4x$ in the 1st member, and transposing 370 to the second we have

$$-4x = 350 - 370,$$

and reducing it according to the rule in art. 19 there will result

$$-4x = -20.$$

But as in the preceding article the sign $-$ which affects the quantity $4x$, has been avoided, by transposing this quantity to the other member, and that in like manner, the quantity $350 - 370$ becomes by transposition $370 - 350$; and since a quantity, by being thus transferred from one member to the other changes its sign (Nos. 4 and 18), it is evident, that we may come to the same result, by simply changing the signs of the quantities $-4x + 350 - 370$, which gives

$$4x = -350 + 370$$

or

$$4x = 370 - 350$$

which equation is the same as

$$370 - 350 = 4x$$

or we might change the signs after reduction, the equation

$$-4x = -20$$

becomes as above

$$4x = 20.$$

Hence it follows, that *we may transpose indifferently, to one member or to the other, all the terms involving the unknown quantity, observing only to change the signs of all the terms in both members in the result, when the unknown quantity has the sign —.*

LVIII. Before giving by means of letters, a general solution of the 18th problem, we will examine a particular case. Supposing that the first sum received by the journeyman to be Rs. 2 : 14, or 46 anas, and the second Rs. 1 : 14, or 30 anas, the other circumstances of the question remaining the same; the equation will then be

$$12x + 7y = 46$$
$$8x + 5y = 30$$

the 1st gives

$$y = \frac{46 - 12 \times x}{7}$$

substituting this value of y in the second equation

$$8x + 5\left(\frac{46 - 12 \times x}{7}\right) = 30$$

or $$8x + \frac{230 - 60x}{7} = 30$$

and the denominator being made to disappear, by multiplying every term by 7,

$$56x + 230 - 60x = 210$$

or $$56x - 60x = 210 - 230$$

or $$-4x = -20$$

and the signs being changed agreeably to what has just been remarked,

$$4x = 20$$

$$x = 5.$$

Substituting this value of x in the expression of y, we obtain

$$y = \frac{46 - 60}{7}$$

and by reduction

$$y = -\frac{14}{7}.$$

Now how is the sign —, which affects the insulated quantity 14 to be interpreted? We may easily conceive what signifies the assemblage of two quantities separated from each other by the sign —, and when the quantity to be subtracted is less than that from which it is to be taken; but how can we subtract a quantity when it is not connected with another in the member in which it is found? To explain this kind of paradox, it is best to go back to the equations, which express the conditions of the question; for the nearer we approach to the enunciation, the closer shall we bring the circumstances which have given rise to the present uncertainty.

Resuming the equation

$$12x + 7y = 46$$

substituting for x its value 5, and it becomes

$$60 + 7y = 46.$$

The sole inspection of this equation presents an absurdity. Indeed, it is impossible to form the number 46 by adding something to the number 60 which exceeds it already.

Taking the second equation

$$8x + 5y = 30$$

and substituting 5 for x, we find

$$40 + 5y = 30$$

the same absurdity as before, since agreeably to this equation, the number 30 is formed by adding something to the number 40 which is already greater.

But the quantities $12x$ or 60, in the first equation, and $8x$ or 40, in the second, express what the laborer earned by his own work, the quantities $7y$ and $5y$ represent the earnings attributed to his children; whilst the sum 46 and 30 indicate the sums given for the common wages of the three; we can now see in what consists the absurdity.

Agreeably to the question, the man earned more by himself alone than he did when aided by his two boys; the money therefore attributed to the work of his two boys cannot possibly augment the pay of the man.

But if, instead of considering the allowance made to his two sons as a gain, we regard it as a charge placed on account of the laborer, then we must subtract that charge from the wages of the man and the equations would no longer involve a contradiction, as they would become

$$60 - 7y = 46$$

$$40 - 5y = 30.$$

We deduce from each of these two equations

$$y = 2$$

whence we conclude, that if the man earned 5 anas per day, his two sons occasioned him an expence of 2 anas, which may thus be verified.

For 12 days of work, the man receives

5 anas $\times$ 12 $=$ 60 anas;

the expence of his two children for 7 days, is

2 anas $\times$ 7 $=$ 14 anas;

or $60 - 14 = 46$ anas, which remain.

For 8 days work, the man gets

5 anas $\times$ 8 $=$ 40 anas

the expence of his two boys during 5 days, is

2 anas $\times$ 5 $=$ 10 anas

$40 - 10 = 30$ anas, remaining.

It is very clear then that in order to render the proposed problem in No. 56 with its conditions possible, we must substitute the following:

A laborer working at a Baboo's house during 12 days, having with him during the first 7 days his two sons, who occasioned him an expence, received Rs. 2 : 14, being afterwards employed during eight days on the same conditions, having his sons with him during 5 days, of whom he had again to bear the expences, he received R. 1 : 14. It is required to know how much he earned per day and what was the sum charged him for the daily expenses of his two sons?

Calling x the daily wages of the laborer and y the daily expence of his two boys, the equations of the problem will evidently be

$$12\,x - 7\,y = 46$$
$$8\,x - 5\,y = 30$$

which being resolved after the manner of those in art. 56, these equations will give

$$x = 5 \text{ anas}, \qquad y = 2 \text{ anas}.$$

LIX. In every case, where we find for the value of the unknown quantity, a number affected with the sign —, the enunciation of the question may be rectified in a manner analogous to the preceding, by examining, with care, what that quantity is, amongst those which are additive in the first equation, which must be subtractive in the second; but algebra supersedes the use of every inquiry of this kind, when we have learnt to make a proper use of expressions affected by the sign —; for these expressions, being deduced from the equations of the problem, must *satisfy* those equations; viz. in subjecting them to the operations indicated in the equation, the first number of the equation must be found to have the same value as the second. Thus the expression $\frac{-14}{7}$ drawn from the equations

$$12\,x + 7\,y = 46$$
$$8\,x + 5\,y = 30$$

must conjointly with the value of $x = 5$, as deduced from these same equations, verify them both.

The substitution of the value of x, gives in the first place,

$$60 + 7\,y = 46$$
$$40 + 5\,y = 30$$

There remains to make the substitution of $\frac{-14}{7} = -2$ instead of y, having regard to the sign — prefixed to the numerator of the fraction, we find,

$$7 \times y = 7 \times -2 = -14$$
$$5 \times y = 5 \times -2 = -10$$

Hence the equations

$$60 + 7\,y = 46 \text{ and } 40 + 5\,y = 30$$

become respectively

$$60 - 14 = 46 \text{ and } 40 - 10 = 30$$

are now verified, not by adding the two parts of the first member, but in reality by subtracting the second from the first, as was done above, after considering the proper import of the equations.

LX. The data in the problem of No. 58, do not admit of a solution in the sense in which it is first enunciated, viz. by addition or regarding the presence of the two boys an accession to the wages of the laborer; neither does the second enunciation consist with the data of the problem in art. 56; indeed, if in this case y is considered as expressing a deduction, the equations thus obtained

$$12\,x - 7\,y = 74$$
$$8\,x - 5\,y = 50$$

would give

$$x = 5 \text{ and } y = \frac{-14}{7}$$

and the substitution of the value of x, will immediately change these equations to

$$60 - 7\,y = 74$$
$$40 - 5\,y = 50.$$

The absurdity of these results is precisely contrary to that of the results in art. 58, since it relates to remainders greater than the number 60 and 40, from which the quantities $7\,y$ and $5\,y$ are to be subtracted.

The sign $-$, which belongs to the expression of y, does not only imply an absurdity, it also rectifies it; for, agreeably to the rules for the signs,

$$\frac{-14}{7} = -2$$

and $-7 \times -2 = +14$

$$-5 \times -2 = +10.$$

By this means, the equations

$$60 - 7\,y = 74, \text{ and } 40 - 5\,y = 50$$

become

$$60 + 14 = 74, \qquad 40 + 10\,y = 50$$

and are verified by addition; and consequently the quantities $-7y$ and $-5y$ transformed into $+14$ and $+10$, instead of expressing expenses, incurred by the laborer, are regarded as a real gain: we are brought back again then in this case, to the true enunciation of the question.

LXI. From the preceding examples we learn, that *in the enunciations of problems of the 1st degree, there may be certain contradictions, which algebra does not only indicate, but points out also how they may be reconciled, by rendering subtractive certain quantities which had been regarded as additive, or additive certain quantities which had been regarded as subtractive, or by giving the unknown quantities values affected with the sign* —.

It is this we are to understand, by the common expression, that the values affected by the sign —, which are called *negative solutions*, resolve the question in which they are found in a contrary sense to its enunciation.

It follows from this, that we ought to regard as but one single question, those of which the enunciations are connected together in such a manner, that the solutions, which satisfy one of the enunciations will, by a mere change of sign, satisfy the other.

LXII. Since negative quantities resolve in a certain sense the problems from which they are derived, it is proper to inquire a little more particularly into the use of these quantities, and to ascertain once for all, the manner of performing operations in which they are concerned.

We have already made use of the rule for the signs, which had been previously determined for each of the fundamental operations; but the rules have not been demonstrated for insulated quantities. In the case of subtraction, for example, we have supposed that from a, the expression $b - c$, was to be subtracted, in which the negative quantity $-c$, was preceded by a positive quantity b (20.)

It would be simple enough to reduce $b - c$ to $-c$ by making $b = 0$, which would change the result into $a + c$; but the reasoning employed in art. 20 supposes the existence of the quantity b, does therefore not seem to embrace this case. The theory of negative quantities being at the same time one of the most important and most difficult in algebra, it should be established upon a sure basis. To arrive at this it is necessary to go back to the origin of negative quantities.

The greatest subtraction, that can be made from a quantity, is to subtract it from itself; in this case we have zero for a

remainder; thus $a - a = 0$. But when the quantity to be subtracted exceeds that from which it is to be taken, we cannot subtract it entirely; we can only make a reduction of the quantity to be subtracted, equal to the quantity from which it was to be taken. When, for example, it is required to subtract 5 from 3, or when we have the quantity $3 - 5$, we decompose 5 into two parts, 3 and 2, of which the successive subtraction will amount to that of 5, we may write then instead of $3 - 5$, the expression $3 - 3 - 2$, which is reduced to -2. The sign $-$, which precedes 2, shews what is necessary to complete the subtraction; so that, if we had added 2 to the first of the quantities, we should have had $3 + 2 - 5$ or zero. We express then, with the help of algebraic signs, the idea that is to be attached to a negative quantity $-a$, by forming the equation $a - a = 0$, or by regarding the expressions $a - a$, $b - b$ as symbols signifying zero.

This being supposed, it will easily be understood that if we add to any quantity whatever, as a, the symbol $b - b$, which in reality is but zero, the value of that quantity a, is not in the least altered, so that the expression $a + b - b$ is only another way of writing the quantity a; which is however evident, since the terms $+b$ and $-b$ destroy each other.

But by this change of form, the quantities $+b$ and $-b$ have entered into the composition of a, it is obvious then that to subtract any one of these quantities, it suffices to efface it. If it were $+b$ that is to be subtracted from this quantity, there remains $a - b$; if on the other hand it were $-b$, that is to be subtracted, we have only to efface this last quantity, and there will remain $a + b$ as may be inferred from art. 20.

With regard to multiplication we observe, that the product of $a - a$ by $+b$ must be $ab - ab$, because the multiplicand $a - a$ being equal to zero, the product must also be equal to zero, the first term being ab the second must necessarily be $-ab$ in order to destroy the first.

We infer from this, that $-a$, multiplied by $+b$, must produce $-ab$. By multiplying a by $b - b$, we have still $ab - ab$, because the multiplier $b - b$ being equal to zero, the product must also be zero; the second term must consequently be $-ab$ in order to destroy the first $+ab$. Whence $+a$, multiplied by $-b$, must produce $-ab$.

Lastly, if we have to multiply $-a$ by $b - b$, the first term of the product being, from what has just been proved $-ab$, the second term must necessarily be $+ab$, since the product must be nothing when the multiplier is nothing.

Whence $-a$ multiplied by $-b$, must give $+ab$. By collecting these results together, we may deduce from them the same rules as those in art. 31.

As the sign of the quotient, combined with that of the divisor according to the rules for multiplication, must re-produce the sign of the dividend, we infer from what has just been said, that the rule for the signs given in art. 42 corresponds with the present case, and that consequently, *simple quantities when they are found insulated, are combined with respect to their signs, in the same manner, as when they form a part of polynomials.*

LXIII. Agreeably to these remarks, we may always, when we meet with negative values, go back to the true enunciation of the question resolved, by seeking in what manner these values will satisfy the equations of the proposed problem ; this will be confirmed by the following example, which relates to numbers of a different kind from those of the question in art. 56.

LXIV. *Two couriers set out at the same time to meet each other from two towns, of which the distance is given ; we know also how many miles each courier travels per hour. It is required to find at what point of the route between the two cities they will meet?*

In order to render the circumstances of the question more evident, we draw a line

$$A \text{—————————————} R \text{——————} B$$

in which the points A and B represent the places of departure of the two couriers.

We shall express as usual the given and sought quantities of the problem by small letters.

a, the number of miles, per hour, which the courier from A travels.

b, the number of miles per hour, which the courier from B travels.

c, the distance in miles of the points of departure A and B.

The letter R represents the point where the two couriers meet.

Let x, be the distance $A\,R$ passed over by the first courier, and y, be the distance $B\,R$ passed over by the second, and seeing that

$$A\,R + B\,R = A\,B$$

we have the equation

$$x + y = c.$$

Considering that the distances x and y are gone over in the same time, it is evident that the first courier who travels a

number of a miles in an hour, will employ to pass over the distance x, a time expressed by $\frac{x}{a}$*.

The second courier who travels b miles in an hour, will employ in passing over the distance y, a time expressed by $\frac{y}{b}$.

We have then the equation

$$\frac{x}{a} = \frac{y}{b}.$$

The two equations of the question will consequently be,

$$x + y = c$$

$$\frac{x}{a} = \frac{y}{b}.$$

Multiplying both members of the second equation by a, to have x alone on one side,

$$x = \frac{ay}{b}$$

substituting this value of x into the first equation, it will become

$$\frac{ay}{b} + y = c$$

from which we deduce

$$ay + by = bc \text{ or } y\,(a + b) = bc$$

whence $$y = \frac{bc}{a + b}.$$

Substituting this value of y into the expression for the value of x, we obtain

$$x = \frac{a}{b} \times \frac{bc}{a + b} = \frac{abc}{b\,(a + b)}$$

or lastly

$$x = \frac{ac}{a + b}.$$

As no — sign enters into the values of x and y, it is evident that whatever numbers are taken for a, b and c, we shall always

* Supposing, to fix the ideas, that a courier travels at the rate of 12 miles an hour, then the time he will require to go over any space, say 48 miles for example, must be expressed by $\frac{48}{12} = 4$. Here 48 stands for x; and 12 for a.

—R- -B

find x and y with the sign $+$, and consequently the proposed problem will be resolved in the precise sense of the enunciation. Indeed it will easily be perceived, that in every case where the two couriers set off in the same time and travel toward each other, they must necessarily meet.

LXV. Let us suppose now, that *the two couriers setting out at the same time proceed in the same direction, the courier who sets out from A is running after the one that sets out from B, who is travelling towards a point C, beyond B with respect to A.*

-B- -R-

It is evident that in this case the courier who starts from A can only meet the courier who sets off from B, unless he travels faster than the last, the point of meeting, R, can no longer be between A and B, but must be beyond B, with respect to A.

Having the same data as before, and observing that in this case

$$AR - BR = AB$$

we have the equation

$$x - y = c.$$

The second equation

$$\frac{x}{a} \quad \frac{y}{b}$$

expressing only the equality of the times employed by the couriers in passing over the distances AR and BR, undergoes no alteration.

The two above equations, being resolved like the preceding ones give,

$$x - \frac{ay}{b}$$

$$\frac{ay}{b} - y = c, \text{ or } ay - by = bc$$

$$\therefore y = \frac{bc}{a-b}$$

$$\therefore x = \frac{a}{b} \times \frac{bc}{a-b} = \frac{abc}{b(a-b)}$$

lastly $$x = \frac{ac}{a-b}.$$

$$A\text{———}B\text{———}R\text{——}C$$

Here the values of x and y will only be positive if a is taken greater than b; viz. in supposing the courier from A travelling at a greater speed than the other.

If for example, we make

$$a = 20 \text{ and } b = 10$$

then

$$x = \frac{20.c}{20-10} = \frac{20.c}{10} = 2c$$

$$y = \frac{10.c}{20-10} = \frac{10.c}{10} = c.$$

Whence it follows, that the point of their meeting R, is at twice the distance from A, than from B, or that $AR = 2\ AB$.

Let us now suppose a smaller than b, and take for example,

$$a = 10 \text{ and } b = 20$$

we find

$$x = \frac{10.c}{10-20} = \frac{10.c}{-10} = -c$$

and for

$$y = \frac{20.c}{10-20} = \frac{20.c}{-10} = -2c.$$

These values being affected with the — sign, make it obvious that the question cannot be resolved in the sense in which it is enunciated; indeed it is absurd to suppose that the courier starting from the point A, travelling only at the rate of 10 miles an hour, should ever overtake the courier setting out from B, travelling at the rate of 20 miles, and who is in advance of the first.

LXVI. These same values resolve nevertheless the question in a certain sense; for, by substituting them in the equations

$$x - y = c$$

$$\frac{x}{a} = \frac{y}{b}$$

we have by the rule for the signs

$$-c + 2c = c$$

and

$$\frac{-c}{10} = -\frac{2c}{20},$$

equations which are satisfied; for by making the reductions that present themselves, the 1st member becomes equal to that of the second, and observing the sign of the terms which

-B- -R-

compose the first, we shall see how the enunciation of the question ought to be modified to do away the absurdity.

Indeed, it is the distance c corresponding to x, passed over by the first courier, which is in reality subtracted from the distance $2c$, corresponding to y, and passed over by the second courier; it is then just as if we had changed x into y and y into x, and had supposed that the courier setting off from the point B had run after the other.

This change, in the enunciation produces also a change in the direction of the routes of the couriers; they are no longer travelling towards the point C, but in an opposite manner towards the point C', as represented in the following diagram:

$$C' \ldots R' \ldots A \text{———} B \text{———} R \text{——} C$$

and their point of meeting is in R'. Thence results

$$BR' - AR' = AB$$

which gives

$$y - x = c$$

and we have as before

$$\frac{x}{a} = \frac{y}{b}$$

or $$x = \frac{ay}{b}$$

and $$y - \frac{ay}{b} = c \text{ or } by - ay = bc \quad \therefore y = \frac{bc}{b-a}$$

and by substituting the numerical value $y = \frac{20\,c}{10} = 2\,c$

$$x = \frac{a}{b} \times \frac{bc}{b-a} = \frac{ac}{b-a},$$

and by the same substitution

$$x = \frac{10.c}{20-10} = c$$

positive values, which resolve the question in the precise sense in which it is enunciated.

LXVII. The question presents a case in which it is in every sense absurd. This happens when we suppose the two couriers to travel at the same rate; it is evident, that they can never meet in whatever direction we suppose them to go, since they preserve constantly the distance of their points of depar-

ture. This absurdity, which no modification in the enunciation can remove, is very conspicuous in the equation

$$\frac{x}{a} = \frac{y}{b}$$

and since $a = b$, becomes

$$\frac{x}{a} = \frac{y}{a}$$

which gives $x = y.$

Thus the equation $x - y = c$ reduces itself into

$$x - x = c \quad \text{or} \quad o = c$$

a very absurd result, since it supposes a given positive quantity c, to be nothing.

LXVIII. This absurdity shews itself in a very singular way in the values of the unknown quantities,

$$x = \frac{ac}{a-b} \qquad y = \frac{bc}{a-b}$$

their denominator $a - b$ becoming o when $a = b$;

$$x = \frac{ac}{o} \qquad y = \frac{bc}{o}.$$

We do not easily perceive what may be the quotient of a division when the divisor is zero; we see merely that if we consider b very nearly equal to a, the values of x and of y would become very great*. To be convinced of this, we need only take

$$a = 10 \text{ miles} \qquad b = 9.5 \text{ miles}$$

$$\text{then } x = \frac{10.\ c}{.5} = 20\ c$$

* To give another example, let the number 36 be successively divided by 36, 18, 9, 6, 3, 2, 1, 0.1, 0.01, 0.001, &c. as:

$\frac{36}{36} = 1$; $\frac{36}{18} = 2$; $\frac{36}{9} = 4$; $\frac{36}{6} = 6$; $\frac{36}{3} = 12$; $\frac{36}{2} = 18$; $\frac{36}{1} = 36$;

$\frac{36}{0.1} = 360$; $\frac{36}{0.01} = 3600$; $\frac{36}{0.001} = 36000$, &c. from this it is also very obvious, that the less the value of the divisor, the greater the quotient.

let b become greater or nearer to a, a remaining constant, $b = 9.9$

$$\text{then } x = \frac{10\ c}{0.1} = 100\ c$$

let b approach still nearer, as $b = 9.95$

$$\text{then } x \quad \frac{10\ c}{.05} = 500\ c.$$

Let b become 9.99999999

$$\text{then } x = \frac{10\ c}{0.00000001} = 1000000000\ c;$$

it is then evident that the divisor diminishes in proportion as the difference of the numbers a and b decreases, and that we obtain values more and more increased in magnitude.

But as a quantity, however minute, can never be taken for zero, it follows that however small we make the difference of the two numbers represented by a and by b, and however great may be the consequent values of x and of y, we can never attain to those which answer to the case where $a = b$.

These last which cannot be represented by any number, however great we suppose it, are termed *infinite;* and every expression of the form, $\frac{m}{o}$ of which the denominator is zero, is regarded as the symbol of *infinity*.

This example shews, that mathematical *infinity* is a negative idea, since we only can attain it by the impossibility of assigning a quantity that can solve the question.

We may ask here, how the values

$$x = \frac{ac}{o} \qquad y = \frac{bc}{o}$$

can satisfy the proposed equation; for it is an essential characteristic of algebra, that the symbols of the values of the unknown quantities whatever they may be, when subjected to the operations indicated upon these unknown quantities, shall satisfy the equations of the problem.

Substituting them in the equations

$$x - y = c$$

and

$$\frac{x}{a} = \frac{y}{b}$$

answering to the case where $a = b$, we obtain by the first,

$$\frac{ac}{o} - \frac{bc}{o} = a,$$

or $\frac{ac - bc}{} = a$ or $ab - bc = a \times o$

and lastly $o = o$ since $a \times o = o$.

The second equation $\frac{x}{a} = \frac{y}{b}$ gives under the same circumstances, always supposing $a = b$,

$$\frac{ac}{o \times a} = \frac{ac}{o \times a};$$

the two members of each equation becoming equal, the equations are satisfied.

It remains still to be explained how the expression $\frac{ac}{o}$, removes the absurdity of the result found in art. 67. For this purpose, let both members of the equation

be divided by x $\qquad x - y = c$

then $\qquad 1 - \frac{y}{x} = \frac{c}{x}$

and since the equation

$$\frac{x}{a} = \frac{y}{a}$$

gives $x = y$, the first will become

$$1 - 1 = \frac{c}{x} \quad \text{or} \quad o = \frac{c}{x}.$$

The error lies here in the quantity $\frac{c}{x}$, by which the second member exceeds the first; but this error diminishes in proportion to the assumed magnitude of x. It is then with reason that algebra assigns for x an expression, which cannot be represented by any number, however great, but which, proceeding in the order of numbers continually increasing, indicates in what manner we may reduce the error of the supposition.

LXIX. If the couriers were travelling at the same rate, and in the same direction, had started from the same point, their junction would not take place, at any particular point since they would be together through the whole extent of their journey. It may be worth while to examine how this circumstance is represented by the values which the unknown quantities of x and y assume in this case.

the points A and B being coincident, we obtain on this supposition $c = o$ and as before $a = b$; by this substitution into the values of x and y, we obtain

$$x - \frac{o \times a}{o} = o \qquad\qquad y = \frac{o \times b}{o} = o.$$

In order to interpret these values, which indicate a division, in which the dividend, and the divisor are each nothing, it is necessary to go back to the equations of the question. The first becoming

$$x - y = o \quad \text{gives} \quad x = y$$

and substituting this value in the second equation, which is in this case

$$\frac{x}{a} = \frac{y}{a} \quad \text{becomes} \quad \frac{y}{a} = \frac{y}{a}.$$

The last equation having both its members *identical*, viz: both composed of the same terms, with the same sign, is verified, whatever value be assigned to y, but we can never determine it from this unknown quantity. Besides it is evident that the equation

$$\frac{x}{a} = \frac{y}{a} \text{ gives } x = y,$$

and consequently nothing more than the first equation*. The only result from either the one or the other, is that the couriers are always together, since the distances x and y from the point A are equal; their value in other respects remains indeterminate. The expression $\frac{o}{o}$ then, is here a symbol of an indeterminate quantity; we say here, for there are cases where it is not; but then, the expression has not the same origin as the preceding.

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* For the sake of conciseness, analysts apply the epithet *identic* even to equations. $\frac{y}{a} = \frac{y}{a}$ is an identical equation, $5 - 3x = 5 - 3x$ is another; and when two equations express only the same thing, we say that these equation also are identical.

LXX. This will be better understood by an example, let there be,

$$\frac{a\,(a^2 - b^2)}{b\,(a - b)}.$$

This quantity becomes $\frac{o}{o}$ in its present form, when $a = b$; but if we reduce it first to its most simple expression, by suppressing the factor $a - b$, common to the numerator and the denominator, we find

$$\frac{a\,(a + b)}{b}$$

which gives $2\,a$, when $a = b$.

This is not the case with the values of x and y, found in the preceding article, for they are not susceptible of being reduced to a more simple expression.

It follows, from what we have just said, that when we meet with an expression that becomes $\frac{o}{o}$, it is necessary, before pronouncing on its value, to examine if the numerator and denominator have not some common factor, which becoming nothing (as $a - b$ when $a = b$), renders the two terms equal at the same time to zero, and by suppressing this common factor, the true value of the proposed equation is obtained. There are notwithstanding some cases which elude this method, but the limits of the present work, do not permit to note this *analytical fact*, which cannot well be rendered intelligible by any elementary part of algebra, the general processes to obtain the true value of the quantities that become $\frac{o}{o}$, are satisfactorily demonstrated by the differential calculus*.

LXXI. It is evident from what precedes, that *algebraic solutions either answer completely to the condition of a problem, when possible, or they indicate a modification to be made in the enunciation, when the things given imply contradictions which cannot be reconciled; or lastly, they make known an absolute impossibility, when there is no method of resolving with the same data, a question analogous in a particular sense to the one proposed.*

* See Lacroix's Treatise of the Differential and Integrate Calculus, 3 vols., quarto, or in the octavo vol. Elementary Treatise, page 341.

LXXII. In the solution of the different cases of the preceding question it is necessary to remark, that the change of the signs of the unknown quantities x and y corresponds to a change in the direction of the routes represented by the unknown quantities. When the unknown quantity y was counted from B towards A, it had in the equation

$$x + y = c$$

the sign $+$, it takes the sign $-$ for the second case, when the motion is in the opposite direction, from B towards C, (65) where we had for the first equation

$$x - y = c.$$

By changing the sign in the second equation,

$$\frac{x}{a} = \frac{y}{b}$$

we have

$$\frac{x}{a} = \frac{-y}{b}$$

a result which differs from that given in art. 65; but it should be observed that the journey y, being composed of multiples of the space b which the courier from B passes over in one hour, which space having the same direction as the space y, must be supposed to have the same sign, and consequently assume the sign $-$, when $-$ is applied to y; we consequently have

$$\frac{x}{a} = \frac{-y}{-b} \quad \text{or} \quad \frac{x}{a} = \frac{y}{b}$$

A simple change of sign then is sufficient to comprehend the second case of the question in that of the first; it is thus that algebra gives in the same time the solution of several analogous questions.

The problem in art. 56, offers a very striking example of this. It was supposed in that article, that the father owed the son a sum d; if we could solve the question on the contrary hypothesis, that is, by supposing that the son owed the father the sum d, it would be sufficient to change the sign of d, in the value of x, we shall thus have:

$$x = \frac{bc - d}{a + b}$$

if, lastly, we suppose father and son to be quit towards each other, we have but to consider $d = o$, the equation would then be

$$x = \frac{bc}{a + b}.$$

Nothing can be simpler than to verify these two solutions by putting again the problem into equation for each of the cases, which we have enunciated.

LXXIII. It was only to preserve an analogy between the problems of Nos. 56 and 64, that we have employed in the second, two unknown quantities. The one as well as the other might have been resolved with one unknown quantity; for when we say that the laborer received Rs. 4 : 10 anas for 12 days' work performed by himself, including the 7 days' work by his two sons, it follows that if we denote by y the daily wages of his two sons and subtract from 74 anas, 7 y, there remains 74 — 7 y for 12 days' labour of the man; whence we infer that he earned $\frac{74 - 7y}{12}$ per day.

By a similar calculation for the 8 days' service, we find that he earned $\frac{50 - 5y}{8}$ per day.

Equalizing the two quantities, we form the equation

$$\frac{74 - 7y}{12} = \frac{50 - 5y}{8}.$$

Also in the question of art. 64,

$$A \text{———} R \text{———} B$$

if x represents the route AR of the courier from A, then $BR = AB - AR = c - x$ would be that of the courier who set off from B towards A. These two distances being gone over in the same time by the two couriers, whose rate of travelling per hour in miles, is represented by a and b respectively, we have

$$\frac{x}{a} = \frac{c - x}{b}$$

whence $$bx = ac - ax$$ and

$$x = \frac{ac}{a + b}.$$

The difference between the solution which we have now given, and those of articles 56 and 64, consists only in this, that we have formed and resolved the first equation by the aid of ordinary language, without employing algebraic characters, and it is evident, that the more we extend the use of this, the less remains to be done with the other.

LXXIV. A circumstance is sometimes added to the problem in art. 64, which does not render it more difficult.

2. *We suppose that the courier who sets off from* B, *starts a number of* d *hours before the other, who departs from* A.

-*R*- -*B*.

It is obvious that this amounts only to a change of the point of departure of the first; for if he travelled a number of b miles per hour, he would pass over the space $BC = bd$ in d hours, and would be at a point C, when the other courier set off from A, so that the interval of the points of departure would be

$$AC = AB - BC = c - bd.$$

By writing then $c - bd$ instead of c in the equation of the preceding article, we have

$$\frac{x}{a} = \frac{c - bd - x}{b}$$

or

$$x \quad \frac{ac - abd}{a + b}.$$

If the couriers were travelling in the same direction, adding the same circumstance as in the preceding example,

A- -*B*- -*R*.

The interval of the points of departure would be

$$AC = AB + BC = c + bd$$

and the distance passed over by the courier from the point A, to the place R where he will overtake the courier from B, would be AR, while that of the other courier would be

$$CR = AR - AC$$

we have then the equation

$$\frac{x}{a} = \frac{x - c - bd}{b}$$

or

$$\frac{ac + abd}{a - b}.$$

LXXV. Enunciated in this manner, the problem presents a case, in which the interpretation of the negative value found for x is attended with some difficulty; it is by making the couriers travel in opposite directions, we assume for d a value, such that the space BC represented by bd, becomes greater than c, which represents AB.

$$C\ldots\ldots R\ldots\ldots A\cdot$$

the courier then from B arrives at C on the other side of A, at the moment when the courier from A starts towards B; it is therefore absurd to suppose that the couriers can ever meet.

If, for example, we should take

$c = 400$ miles; $a = 12$ miles; $b = 8$ miles; $d = 60$ hours, the value of bd would then be $= 480$ miles; thus the courier from B would be at the point C, which is 80 miles beyond A with respect to the point B, at the time the courier from A sets out; we should thus find as in the preceding article, viz.

$$x = \frac{ac - abd}{a + b}$$

$$x = \frac{12 \times 400 - 12 \times 8 \times 60}{12 + 8} = \frac{3 \times 400 - 12 \times 2 \times 60}{3 + 2}$$

$$= \frac{1200 - 1440}{5} = \frac{-240}{5} = -48.$$

The point of meeting of the couriers would thus be in R which is 48 miles beyond A, but between A and C, although it appears that the courier from B, being supposed to continue his journey beyond the point C, could not be overtaken by the other courier but after having passed this point.

In order to understand the question resolved in this sense, let us substitute in the 1st equation of art. 74, instead of x the negative number $-m$, and the equation becomes

$$\frac{-m}{a} = \frac{c - bd + m}{b}$$

or by changing the signs in both members

$$\frac{m}{a} = \frac{bd - c - m}{b}.$$

We now see that the distance passed over by the courier from the point B, is

$$C\ldots\ldots R\ldots\ldots A\text{———————}B$$

$bd - c - m$, or what remains of the distance BC, after subtracting from it AB and AR, that is CR, and that $AC = bd - c$. This is just what would take place if the second courier had started immediately from the point C, where he is at the time of departure of the first; but as they travel in opposite directions, their meeting must necessarily take place between A and C. This case then is similar to the first of those of art. 74, where it is sufficient to change $c - bd$ into $bd - c$, in order to obtain the value which m has agreeably to the above equation.

LXXVI. The problem of art. 56, taken in its most enlarged sense may be enunciated as follows:

3. *A laborer was employed a number of* a *days in a house, and having with him his two sons for a number of* b *days, received a sum* c; *he afterwards was engaged on the same*

NOTE. By leaving out in the enunciation of the problem of art. 65, the idea of a fixed time of departure, and to suppose them to have been travelling from an indefinite time; the question then would be stated thus. *Two couriers travel the same rate in the same direction $CABC$; after each had marched a certain time, one finds himself in A at the instant that the other is in B; their rate of travelling and the distance AB, are known; it is required to ascertain at what point of the route they will meet?*

C'......R'......A————B————R———C

This enunciation leads to the same equation as that of art. 65; but when the continuity of motion is once established, the negative solution admits of an explanation without the necessity of changing the direction of one of the couriers. Indeed, since their motion does not commence at the points A and B, but both, before arriving at these points, are supposed to have been going in the same manner for an indefinite time from C' towards B, it is easy to conceive, that the courier, who at this point is in advance of the other courier, who then is at A who travels slower, must at a certain time have been behind him and overtaken him before reaching the point A. The sign then indicates (as in the application of algebra to geometry) that the distance AR' must be taken in a direction opposite to that of AR which is regarded as positive. The change to be made in the enunciation to render the negative solution positive, is reduced to supposing, that the two couriers must have met before their arrival at the point A, instead of its taking place afterwards.

Indeed in placing the point R' between A and C' instead of between B and C we find $AB = BR' - AR'$; whence results the equation $c = y - x$ instead of $c = x - y$ which we first obtained, and there is no need now of changing the sign of b. The second equation remains $\frac{x^2}{a} = \frac{y}{b}$.

The problem of art. 75, by substituting moveable bodies, for the couriers subjected to a continued motion, commencing from an indefinite time, might be thus enunciated: *Two moveable bodies whose velocities are known, moving uniformly on the same straight line CB, one in the direction BC, the other in that of CB; that which moves in the first direction, is found in B a known number of hours before the other had arrived at A: it is required to ascertain at what point of the indefinite straight line BC their meeting takes place?*

-R'- -B- -A.. R...C.

The solution $x = -48$ miles expresses, that the moveable bodies must have met at the point R' before that which is moving from C towards B, had arrived at the point A, and that the first, moving from B towards C, has arrived at the point C at the instant of time when the other is at the point A.

conditions, a number of d *days; this time he had with him his two sons during a number of* e *days, and he received a sum* f. *It is required to ascertain how much he earned per day, and what was allowed per day to his two sons?*

Let x represent, as before, the laborer's daily wages, and y that of his two sons; for a number of a days he will receive ax, for the number of b days his two boys will get by, so that we have the equation

$$ax + by = c,$$

for the number of d days, he has dx, and for the number of e days his two sons get ey, we have thus, the second equation

$$dx + ey = f.$$

These are the two general equations of the equation.

From the first we obtain

$$x \quad c - by$$

Multiplying this value of x by d, to substitute it in place of dx in the second equation, we get

$$\frac{dc - dby}{a} + ey = f$$

and by clearing this quantity of the partial denominator, we have

$$dc - dby + aey = af$$

or
$$y\,(ae - db) = af - cd$$

$$\therefore \qquad y = \frac{af - cd}{ae - bd}$$

The position assigned to the point R, verifies itself by observing that there results from it $AC = BC - AB = bd - c$, instead of $c + bd$, which we obtained at first,

consequently $$\frac{x}{a} = \frac{bd - c - x}{b}$$

which gives $$x = 48.$$

In this manner there is no change to be made in the direction of the motion, indeed the material circumstances of the problem are changed, which proves, as we said before, that there are several physical questions corresponding to the same mathematical relations: but the enunciations here given, have the advantage of not breaking the laws of continuity, and this is derived from the consideration of lines which represent in the most simple manner the circumstances of the change of sign in magnitudes.

Knowing the value of y, if we substitute it instead of y in the expression for x, this last will be known,

$$x = \frac{c - b\left(\frac{af - cd}{ae - bd}\right)}{a}.$$

To simplify this expression, we must in the first place perform the multiplication by b, which is only indicated to be done on the quantity

$$\frac{af - cd}{ae - bd},$$

consequently $x = \dfrac{c - \dfrac{abf - bcd}{ae - bd}}{a}$

in order to reduce c to the denominator of the fraction which accompanies it, and to perform the subtraction as indicated, we obtain

$$x = \frac{\dfrac{ace - bcd - abf + bcd}{ae - bd}}{a}$$

or by reducing

$$x = \frac{\dfrac{ace - abf}{ae - bd}}{a}*$$

* That there should be no doubt about the sense of these kind of expressions, it is necessary to be attentive to the bar which separates the numerator from the denominator of the fraction. In the expression of $x = \frac{A}{B}$, A represents the dividend, whether integral or fractional, and B the divisor which may also be a whole number or a fraction. So also, the expression $x = \dfrac{\frac{A}{C}}{B}$ signifies, that x is equal to the quotient of the fraction $\frac{A}{C}$ divided by B; and the expression $x = \dfrac{A}{\frac{B}{C}}$ signifies, that x is equal to the quotient arising from A divided by the fraction $\frac{B}{C}$; lastly, the

As the value of a fraction remains unaltered by multiplying or dividing both the numerator and the denominator by the same quantity, we divide both members by a,

then $$x = \frac{ace - abf}{a^2e - abd}$$

but every term of this fraction being affected by a, its value is not altered by suppressing this common factor, we have then

$$x = \frac{ce - bf}{ae - bd}.$$

The values

$$x = \frac{ce - bf}{ae - bd}, \quad \text{and} \quad y = \frac{af - cd}{ae - bd}$$

are applied in the same manner as those which we found before for literal equations, with only one unknown quantity; we substitute in the place of the letters, the particular numbers in the example selected. We shall obtain the results in art. 56, by making

$$a = 12 \quad b = 7 \quad c = 74$$
$$d = 8 \quad e = 5 \quad f = 50$$

or those of art. 58, by making

$$a = 12 \quad b = 7 \quad c = 46$$
$$d = 8 \quad e = 5 \quad f = 30.$$

LXXVII. The values of x and y are not only adapted to the proposed question; they extend also to all those, that lead to two equations of the first degree having two unknown quantities; since it is evident, that these equations are necessarily comprehended in the formulæ

$$ax + by = c$$
$$dx + ey = f$$

provided the letters a, b, d and e express the whole of the given quantities by which the unknown quantities x and y, are

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expression $x = \dfrac{\frac{A}{C}}{\frac{B}{D}}$ denotes that the value of x is equal to the quotient resulting from the fraction $\frac{A}{C}$, divided by the quotient of the fraction $\frac{B}{D}$.

These remarks show the necessity to place the bars conformably to the result which we propose to express.

respectively multiplied and the letters c and f, the whole of the known terms transposed to the second member.

4. *A man as old as the sum of the ages of his five children, has a wife 6 years younger than himself, whose age equals that of the two eldest together with that of the two youngest children, who happened to be twins. The age of the third child was equal to the sum of the ages of the two youngest, the difference between the ages of the eldest and second child was equal to that of the father and mother, and the sum of the ages of the father and mother is less by 31 years, than the second child's age multiplied by 8. Required the age of the man and his wife?*

Let x stand for the age of one of the twin children,
and y for that of the eldest child,
then the age of the father $= 2x + 2x + y + y - 6$
and that of the mother $= 2x + 2y - 6$.
The difference $2x$, is (by hypothesis) $= 6$,

$$\therefore\ x = 3$$

NOTE.—Every question in which there are two unknown quantities and from which we can form two equations of the 1st degree may be solved in three different ways, by *substitution*, by *comparison*, and by *addition* or *subtraction*; each method may in particular cases become preferable to the other two. To give an example

1st equation $ax + by = n$

2nd do. $dx - ey = m$.

By *substitution*.

From the 1st equation, we get

$$x = \frac{n - by}{a}$$

substituting this value of x into the 2nd equation

$$\frac{dn - bdy}{a} - ey = m$$

or $dn - bdy - aey = am$ or $y\,(ae + bd) = dn - am$

$$\therefore\quad y = \frac{dn - am}{ae + bd}$$

by the same process we can obtain the value of x without first finding that of y; as:

from 1st equation $y = \dfrac{n - ax}{b}$, which substituted in the 2nd

$$dx - \frac{en - aex}{b} = m$$

whence $x, = \dfrac{bm + en}{ae + bd}$

and as the sum of the two quantities expressing the ages of the father and mother, must be equal to 8 times the age of the second child together with 31 years, we have

$$6x + 6y - 12 = 8(y - 6) + 31$$

and substituting the value of x found to be 3

$$18 + 6y - 12 = 8y - 48 + 31$$

or $$6 + 48 - 31 = 8y - 6y \text{ or } 23 = 2y$$

$$\therefore y = 11\tfrac{1}{2} \text{ years;}$$

thence the age of the father is known by substituting for x and y their values as

$$12 + 23 - 6 = 29$$

and that of the mother $6 + 23 - 6 = 23$ years.

By *comparison*

from 1st equation $$x = \frac{n - by}{a},$$

and from 2nd ditto $$x = \frac{m + ey}{d}.$$

Equalising the two values of x, we get the equation

$$\frac{n - by}{a} = \frac{m + ey}{d}$$

in which only one unknown quantity y remains; proceeding

$$dn - bdy = am + aey$$

$$\text{or } y\,(ae + bd) = dn - am$$

$$\therefore y = \frac{dn - am}{ae + bd}.$$

The value of x can be obtained by the same process

from 1st equation $$y = \frac{n - ax}{b}$$

from 2nd ditto $$y = \frac{dx - m}{e}$$

$$\therefore \frac{n - ax}{b} = \frac{dx - m}{e}, \text{ or } en - aex = bdx - bm$$

$$\therefore x = \frac{bm + en}{ae + bd}$$

By *subtraction.*

The 1st term of the 1st equation being the product of a by x, and the 1st term of the 2nd equation, the product of d by x, it follows, that if we multiply each term of the 1st equation, by the quantity d, and each term of the 2nd equation, by the quantity a, the first terms of both equations

5. *A baboo had only daughters by his first wife, but all the children of his second wife were boys; if his second wife had borne half a boy more, the boys would only have been half as numerous as the girls. On the baboo's giving away in marriage the third part of the number of his daughters, less one-third part of a daughter, and adopting his three nephews who had become orphans, he had as many girls left as he then had boys. How many daughters and how many sons had the baboo?*

Let x represent the number of boys,
y that of girls,
then by the 1st condition of the question

$$x + \frac{1}{2} = \frac{y}{2}$$

or $$y = 2x + 1$$

and by the 2nd condition

$$y - \left(\frac{y}{3} - \frac{1}{3}\right) = x + 3$$

or $$2y + 1 = 3x + 9$$

and substituting the value of y from the 1st equation, we get an equation with only one unknown quantity

$$2(2x + 1) + 1 = 3x + 9$$

or $$4x + 2 + 1 = 3x +$$

$$\therefore x = 6 \text{ boys and } y = 2x + 1 = 13 \text{ girls.}$$

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must be equal; subtracting one equation from the other, both the first terms will vanish, viz.

the 1st equation multiplied by d, $adx + bdy = dn$

the 2nd ditto multiplied by a, $adx - aey = am$

subtracting the 2nd equation from the 1st $bdy + aey = dn - am$

$$\therefore y = \frac{dn - am}{ae + bd}$$

The value of x can directly be obtained by *addition.*

For, by a similarreason multiplying each term of the 1st equation by the quantity e, and each term of the 2nd equation by the quantity b, that the terms in both equations in which y enters, may become equal,

multiplying then the 1st equation by e $aex + bey = en$

ditto 2nd do. by b $dbx - bey = bm$

by adding the two equations $aex + dbx = en + bm$

$$\therefore x = \frac{bm + en}{ae + bd}.$$

6. *To divide the number 37 into 5 such parts that when these are respectively divided by 3, 4, 5, 6, 7, their quotient shall be in proportion of 5. 6, 7, 8. 9.*

This problem may be solved by one letter only as follows:

Supposing the number is to be divided into x shares, the several parts will then be 3 times $5.x$ for the 1st part

4	do.	$6.x$	2nd ,,
5	do.	$7x$	3rd ,,
6	do.	$8.x$	4th ,,
7	do.	$9x$	5th ,,

But as these five parts ought to make up the given number 37 we have the equation

$$3 \times 5x + 4 \times 6x + 5 \times 7x + 6 \times 8x + 7 \times 9x = 37$$

or $\quad x(15 + 24 + 35 + 48 + 63) = 37$

$$\therefore x = \frac{37}{185} = \frac{1}{5},$$

it is evident that if the 1st part be divided by 3, the 2nd by 4, &c, the quotients, $5x$, $6x$, &c. will be in the proportions required

the 1st part is therefore $15 \times \frac{1}{5} = 3$

the 2nd do. .... $24 \times \frac{1}{5} = 4\frac{4}{5}$

the 3rd do. .... $35 \times \frac{1}{5} = 7$

the 4th do. .... $48 \times \frac{1}{5} = 9\frac{3}{5}$

the 5th do. .... $63 \times \frac{1}{5} = 12\frac{3}{5}$

by addition we get 37 which is the number to be divided.

7. *A baboo bought an elephant and a horse. On selling them he obtained the same sum for each animal, but lost Rs. 624, by the bargain, losing only one per cent. on the elephant, but 34 per cent. on the horse. How much did he pay for each?*

Let x represent the cost of the elephant and y that of the horse;

$x - \frac{x}{100}$ indicates the price he paid for it, less the 100th part of it which he lost, which is the sum he sold it for, and $y - \frac{34y}{100}$

the price he paid for the horse, less 34 times the 100th part its cost or the selling price, and since the amount he received for each is the same, we have

the 1st equation $$x - \frac{x}{100} = y - \frac{34\,y}{100}$$

or $$99\,x = 66\,y$$

$$\therefore\ x = \frac{2}{3}y$$

and by the second condition, viz. that both losses amounted to Rs. 624, we get the second equation.

$$\frac{x}{100} + \frac{34\,y}{100} = 624$$

or $$x + 34\,y = 62400.$$

Substituting for x its value $\frac{2}{3}y$

$$\frac{2}{3}y + 34\,y = 62400$$

or $$y\,(2 + 102) = 187200$$

$\therefore\ y =$ Rs. 1800 the price of the horse.

Substituting this in the equation of $x = \frac{2}{3}y$

we have $x =$ Rs. 1200 the price of the elephant.

8. *I gained on an article 25 per cent, but lost on another 45 per cent.; on the whole I gained a sum* a. *Had I gained on the first article as much as I had lost on the second, and gained on the second as much only as I gained on the first, my profit would have been five times as much. What was the price of each article?*

Be x the first article, and y the second, (the gains can very properly be considered positive, and the losses negative quantities,) then by the 1st condition

$$\frac{25}{100}x - \frac{45}{100}y = a$$

$$\therefore\ x = 4a + \frac{9}{5}y$$

by the 2nd supposition, we get

$$\frac{45\,x}{100} + \frac{25y}{100} = 5\,a$$

$$\therefore\ x = 11\frac{1}{9}\,a - \frac{5}{9}y.$$

By comparison

$$11\frac{1}{9}a - \frac{5}{9}y = 4a + \frac{9}{5}y$$

or

$$7\frac{1}{9}a = y\left(\frac{5}{9} + \frac{9}{5}\right) \text{ multiplying by 45}$$

$$320a = 106y$$

$$y = \frac{160}{53}a$$

taking instead of a, the number $= 53$

then $y = 160$

and $x = 500.$

9. *As there were several claimants for a zemindary, the judge made the following decree: that the 1st claimant should have 600 biggahs and the 12th part of the remaining biggahs; the second claimant should take 1200 biggahs and the 12th of what remained of the zemindary; the third should receive 1800 biggahs and the 12th part of what remained, and so on for the rest of the claimants. After each of the claimants had taken possession of his share, they were surprised to find that the zemindary had been equally divided amongst them. What was the number of biggahs in the zemindary, the share of each, and the number of claimants?*

Let x denote the number of biggahs in the zemindary and y that the share of each claimant, x therefore, is the product of a share y, multiplied by the number of claimants, or $\frac{x}{y}$ denotes the number of claimants,

1st claimant's share $= 600 + \frac{x - 600}{12}$

2nd ditto ditto $= 1200 + \frac{x - 1200 - y}{12}$

3rd ditto ditto $= 1800 + \frac{x - 1800 - 2y}{12}$

4th, &c.

But since the shares of each claimant are equal, their difference must be nothing; subtracting then the share of the 1st from that of the 2nd claimant we have

$$1200 + \frac{x - 1200 - y}{12} - \left(600 + \frac{x - 600}{12}\right) = 0$$

which equation gives

$$\therefore y = 6600$$

substituting this value of y into the expression of the share of the 2nd claimant, we have

$$y, \text{ or } 6600 = 1200 + \frac{x - 1200 - 6600}{12}$$

whence we get $\qquad x = 72600$

and the number of claimants were

$$\frac{x}{y} = \frac{72600}{6600} = 11.$$

It appears therefore that the zemindary consisted of 72600 biggahs, that the share of each claimant was 6600 biggahs, and that there were 11 claimants.

10. *A father leaves his fortune to his children on the following conditions : the* 1*st receives* a *Rupees together with the* n*th part of the remainder ; the* 2*nd received* 2a *Rupees, together with the* n*th part of the remainder, and so on, each succeeding child receives* a *Rupees more together with the* n*th of the remainder, and it is found at last, that they have all received the same amount ?*

Ans. *The fortune* $= (n-1)^2 a.$
Share of each child $= (n-1)\ a.$
Number of children $= \ n-1.$

11. *A boat having wind and current in its favor can go from Calcutta to Barrackpore (*16 *miles) in* 2 *hours, making use of two oars, but on its return from thence, having no wind, and the current in opposition, it requires three oars to return in equal time ; also the boat can reach in eight hours having no oars but the same wind and current in its favor. What is the proportional force of the oar, wind and current ?*

Ans. The force of wind and that of the current are the same, that of an oar is 3 *times stronger than either.*

12. *A man sold a number of fruits to some boys, and received* 12 *Rs. for them, being questioned by another person about the rate he sold his fruits to the boys, he answered ; the rate is equal to the difference between the number of fruits and boys, and if the boys had taken* 6 *fruits more for the same price, the rate would have been half the difference between them and the number of fruits. What is the number of boys, and how many fruits did they take ?*

Ans. 4 *boys and* 6 *fruits.*

13. *Two girls* A *and* B *were going to a market to sell their respective number of eggs, and found that the number of eggs which* A *had, was greater than the number of eggs which* B *had, but* A *sold her eggs at the rate of* 3 *for a pice, and* B *sold half the number of her eggs at the rate of one, and the other*

half at the rate of 3 for a pice. On their return, B found that she had as many pice more than A, as many eggs she had less than A. The next day each of them brought the same number of eggs to market, A sold her eggs again at the same rate of 3 eggs for a pice, and B sold 16 of hers at the rate of 2 for a pice, and the remainder she sold 4 eggs for the pice, when each girl received an equal quantity of pice. How many eggs did each girl bring to market?

Ans. The number of A's *eggs is* 30
The ditto B's *eggs is* 24

14. *A person has two snuff-boxes. If he put* 8 *rupees into the* 1*st, then it is half as valuable as the other. But if he take these* 8 *rupees out of the* 1*st, and put them into the* 2*nd, then the latter is worth three times as much as the former. What is the value of each?*

Ans. The 1*st worth* 24 *Rs., the* 2*nd* 64 *Rs.*

15. A *and* B *possess together a fortune of* 570 *rupees. If* A's *fortune were three times, and* B's 5 *times as great as each really is, then they would have together* 2350 *rupees. How much has each?*

Ans. A 250 *Rs., B* 320 *Rs.*

16. *Find* 2 *numbers of the following properties. When the one is multiplied by* 2, *the other by* 5, *and both products added together, the sum is* = 31; *on the other hand, if the* 1*st be multiplied by* 7, *the* 2*nd by* 4, *and both products added together, we shall obtain* 68.

Ans. The 1*st is* 8, *the* 2*nd is* 3.

17. *If one of* 2 *numbers be multiplied by* a, *the other by* b, *the sum of the products* = k; *but if the* 1*st be multiplied by* a′, *the* 2*nd by* b′, *then the sum of the products* = k′. *How can these numbers be expressed?*

$$Ans. \quad By \ \frac{b'k + bk'}{ab' - a'b}, \quad \frac{ak' - a'k}{ab' - a'b}$$

18. *Two numbers are given by the following data. If the* 1*st be increased by* 4, *it will be* $3\frac{1}{2}$ *times as great as the* 2*nd; but if the* 2*nd be increased by* 8, *then it will be half as great as the* 1*st. What are the numbers?*

Ans. 48 *and* 16.

19. *When the* 1*st of* 2 *numbers is increased by* a, *it becomes* m *times as great as the* 2*nd; but when the* 2*nd is increased by* b, *then it is* n *times as great as the* 1*st. How are these numbers expressed?*

$$Ans. \quad The \ 1st = \frac{a + mb}{mn - 1}; \ the \ 2nd = \frac{b + na}{mn - 1}.$$

20. *Said a man to his father, "How old are we?" "Six years ago," answered the latter, "I was $\frac{1}{3}$ more than 3 times as old as you; but 3 years after this I was so old, that I was obliged to multiply your age by $2\frac{1}{5}$, in order to obtain my own." What is the age of each?*

Ans. The father 36, *the son* 15 *years.*

21. A *and* B *jointly have a fortune of* 9800 *Rs.* A *invests the* 6*th part of his property in business, and* B *the* 5*th part, when each has the same sum remaining. How much has each?*

Ans. A *has* 4800 *Rs.*; B 5000 *Rs.*

22. A *owes* 1200 *Rs.*, B 2550 *Rs.*, *but neither has enough to pay his debts. Lend me, said* A *to* B, *the* 8*th part of your fortune, and I shall be enabled to pay my debts.* B *answered, I can discharge my debts, if you will lend me the* 6*th part of yours. What was the fortune of each?*

Ans. A's *fortune is* 900 *Rs.*, *and that of* B 2400 *Rs.*

23. *A capitalist borrows* 8000 *Rs. on favorable conditions, because he has an opportunity of laying* 23000 *Rs. out to higher interest, and he has an overplus of* 905 *Rs. of interest yearly. Under the same conditions he borrows, on the one hand,* 9400 *Rs. and on the other, lends* 17500 *Rs.; this brings him in an overplus of* $539\frac{1}{2}$ *Rs. in interest yearly. At what rate of interest did he borrow and lend money?*

Ans. At $4\frac{1}{2}$ *and* $5\frac{1}{2}$ *per cent.*

24. *A person has two large pieces of ice, whose weight is required. It is known that* $\frac{4}{9}$*ths of the* 1*st piece weighs* 96 *lbs. less than* $\frac{3}{4}$*ths of the other piece, and that* $\frac{5}{8}$*ths of the other piece weighs exactly as much as* $\frac{4}{9}$*ths of the* 1*st. How much does each of the pieces weigh?*

Ans. The 1*st weighs* 720, *the* 2*nd* 512*lbs.*

25. *A cistern containing* 210 *buckets, may be filled by* 2 *pipes. By an experiment in which the* 1*st was open* 4, *and the* 2*nd* 5 *hours,* 90 *buckets of water were obtained. By another experiment, when the* 1*st was open* 7, *and the other* $3\frac{1}{2}$ *hours,* 126 *buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be full, when the water flows from both pipes at once?*

Ans. The 1*st pipe discharges* 15, *and the* 2*nd* 6 *buckets; it will require* 10 *hours to fill the cistern.*

26. *A person has two kinds of goods,* 8*lbs. of the* 1*st and* 19*lbs. of the* 2*nd cost together* £18. 4*s.* 2*d.*: *farther,* 20*lbs. of the* 1*st and* 16*lbs. of the* 2*nd cost together* £25. 16*s.* 8*d. How much does the lb. of each article cost?*

Ans. 15*s.* 10*d. and* 12*s.* 6*d.*

27. 15 *Silesian and* 33 *Leipzig ells together are equal to*

$39\frac{1}{4}$ Brabant ells; farther, 24 Silesian and 55 Leipzig ells are together equal to 65 Brabant ells. What proportion, according to these conditions, do the Silesian and Leipzig ells bear to the Brabant? farther, what the Silesian to the Leipzig ell? and by how many per cent. do the two last differ from one another?

Ans. The Silesian ell is to the Brabant ell as 5 to 6, the Leipzig to the Brabant as 9 to 11, the Silesian to the Leipzig as 55 to 54; and the Silesian ell is $1\frac{23}{27}$ per cent. longer than the Leipzig.

28. 40 *French miles, when reduced to geographical or German miles, amount to $12\frac{1}{2}$ such miles more than 58 English. Ten French and $26\frac{1}{2}$ English miles, are together equivalent to $11\frac{3}{4}$ German miles. At this rate, what proportion do the French and English bear to the German? And what relation have the French to the English?*

Ans. The French mile is to the German as 3 to 5, the English is to the German as 23 to 106, and the French to the English as 318 to 115.

29. B *has lent out at interest* 12600 *Rs. more than* A, *and obtains* 1 *per cent. more for his money, on which account his interest amounts yearly to* 750 *Rs. more than* A's. C *has lent* 3000 *Rs. more than* A, *and also at* 2 *per cent. higher interest, he obtains for this sum* 360 *Rs. interest more than* A. *How much has each lent, and at what interest?*

Ans. A *has lent* 10000 *Rs.*, B 22600 *Rs.*, C 13000 *Rs.*; A *at* 4 *per cent.*, B *at* 5, *and* C *at* 6.

30. *A company expended a certain sum in an inn, and each the same (to be equally divided amongst them). Had there been 5 persons more, and had each expended 2s. 6d. more, the reckoning would have come to £6. 18s. 10d. more; but had there been 3 persons less, and each expended 1s. 8d. less, then it would have amounted to £3. 8s. 4d. less. How many were there in the company, and what did each spend?*

Ans. The company consisted of 14 persons, and each spent 16s. 8d.

31. *Required to find 2 numbers such, that if one be increased by* a, *the other by* b, *the product of these 2 sums exceeds the product of the 2 numbers themselves by* c; *if on the other hand, the one be increased by* a′, *the other by* b′, *the product of these sums exceeds the product of the numbers themselves by* c′. *How are these numbers expressed?*

Ans. They are $\dfrac{a'c-ac'+aa'(b'-b)}{a'b-ab'}$ *and*

$$\frac{bc'-b'c+bb'(a-a')}{a'b-ab'}.$$

32. *Are the 2 preceding problems included in this one? And what values must be assigned to the letters* a, b, c, a′, b′, c′, *so that the former problems may be solved?*

33. *Not long ago, says a person, the quarter of wheat was* £1, *and the quarter of rye* 17*s.* 6*d. cheaper than they are now; then the price of the wheat was to the price of the rye, as* 10 *to* 7; *their present prices are as* 4 *to* 3. *What is the price of the quarter of each grain?*

Ans. The quarter of wheat costs £3. 10*s. the quarter of rye* £2. 12*s.* 6*d.*

34. *A person has 2 casks, and a certain quantity of wine in each. In order to have an equal quantity in each, he pours as much out of the* 1*st cask into the* 2*nd as it already contains, then again he pours as much out of the second into the* 1*st as it now contains; and lastly, again, as much from the* 1st *into the* 2*nd as there is still remaining in it. At last he has* 16 *quarts of wine in each cask. How many quarts did they contain originally?*

Ans. In the 1*st* 22 *quarts, in the* 2*nd* 10 *quarts.*

35. *When, in the preceding problem, there remain at last* a *quarts in each cask: how many quarts must they have contained originally?*

Ans. The 1*st* $\frac{11}{8}$ a, *the* 2*nd* $\frac{5}{8}$ a *quarts.*

36. *A Wine merchant has 2 kinds of wine. If he mix* 3 *quarts of the best with* 5 *quarts of the worst, he can sell the mixture for* 17*s.* 1*d. a quart; but if he mix* $3\frac{3}{4}$ *quarts of the best with* $7\frac{1}{2}$ *quarts of the worst, then he can sell the quart for* 16*s.* 8*d. exactly. What does each wine cost a quart?*

Ans. The best Wine £1. 3*s.* 4*d. a quart, the worst* 13*s.* 4*d. a quart.*

37. *In general, let* a *quarts of the* 1*st wine mixed with* b *quarts of the* 2*nd, be worth, on an average,* c *shillings; farther, let* f *quarts of the* 1*st mixed with* g *quarts of the* 2*nd, average worth* h *shillings. What does each wine cost a quart?*

Ans. The price of the 1*st is* $\frac{(a+b)\,cg-(f+g)\,bh}{ag-bf}$ *shillings, the price of the* 2*nd is* $\frac{(a+b)\,cf-(f+g)\,ah}{bf-ag}$ *shillings.*

38. 37*lbs. of tin lose* 5*lbs. in water, and* 23 *lbs. of lead lose* 2*lbs. in water; a composition of tin and lead weighing* 120*lbs. loses* 14*lbs. in water. How much does this composition contain of each metal?*

Ans. 74*lbs. of tin, and* 46*lbs. of lead.*

39. *21lbs. of silver lose 2lbs. in water, and 9lbs. of copper lose 1lb. in water. If a composition of silver and copper weighing 148lbs. loses $14\frac{2}{3}$lbs. in water, how much does it contain of each?*

Ans. 112*lbs. of silver, and* 36*lbs. of copper.*

40. *A given piece of metal, which weighs* p *lbs. loses* a *lbs. in water. This piece, however, is composed of 2 other metals, which call* A *and* B; *of these we know, that* p *lbs. of* A *lose* b *lbs. in water, and* p *lbs. of* B *lose* c *lbs. How much does this piece contain of each metal?*

Ans. $\frac{c-a}{c-b}$ p *lbs. of* A, *and* $\frac{a-b}{c-b}$ p *lbs. of* B.

41. *According to Vitruvius, Hiero, king of Syracuse's crown weighed 20lbs. and lost $1\frac{1}{4}$lbs. nearly in water. Let it be assumed, that it consisted of gold and silver only, and that* 19.64*lbs. of gold lose* 1*lb. in water, and* 10.5*lbs. of silver, in like manner, lose* 1*lb. How much gold and how much silver did this crown contain?*

Ans. 14.77*lbs. of gold, and* 5.22*lbs. of silver.*

Is this problem contained in the preceding one?

And what must be here assumed for p, a, b, c?

42. *Lead is* 11.324 *heavier than water; cork only weighs* 0.24 *times as much as water; again fir weighs* 0.45 *times as much as water. A person wishes to combine a piece of lead with a piece of cork, so that he may obtain a body of* 80*lbs. weight, exactly of the same weight as a piece of fir of the same size, (which, consequently, will float.) How much lead and cork must be combined together?*

Ans. 38.14*lbs. of lead with* 41.85*lbs. of cork.*

43. *Two different kinds of matter, one of which is* p, *and the other* p′ *times as heavy as water, are to be so united, that the bodies which arise from their union may on an average be* p″ *times as heavy as water and weigh* 9*lbs. In order to effect this, how many lbs. must be taken of each substance?*

Ans. $\frac{qp\,(p'-p'')}{p''\,(p'-p)}$ *lbs. of the* 1*st, and* $\frac{qp'\,(p''-p)}{p''\,(p'-p)}$ *lbs. of the 2nd substance.*

Between what limits ∴ must p″ *lie, that the problem, as it is here given, may be possible?*

44. *It is required to find 2 numbers whose difference, sum, and product are to one another as the numbers 2, 3, 5, consequently, whose difference is to their sum as 2 to 3, and whose sum is to their product as 3 to 5. What numbers are they?*

Ans. 2 *and* 10.

45. *Find 3 numbers, whose sum is* m, *and whose product is* n *times as great as their difference. What numbers are they?*

Ans. $\frac{2n}{m-1}$ *and* $\frac{2n}{m+1}$.

46. *Let the sum of 2 numbers be* 13, *and the difference of their squares* 39. *What numbers are they?*

Ans. 5 *and* 8.

47. *The sum of 2 numbers is* = a, *the difference of their squares* = b. *What are the numbers?*

Ans. $\frac{a^2+b}{2a}$ *and* $\frac{a^2-b}{2a}$.

48. *The sum of 2 numbers* = a, *the quotient arising from the divisor of the one by the other, is* = b. *Find these numbers?*

Ans. $\frac{a}{b+1}$ *and* $\frac{ab}{b+1}$.

49. *A person was asked his own, his father's, and grandfather's ages. He answered, "My age and my father's amount to* 56 *years, my father's age and my grandfather's to* 100 *years, mine and my grandfather's to* 80 *years." What was the age of each?*

Ans. He is himself 18 *years, his father* 38, *and his grandfather* 62 *years old.*

50. *The sums of* 3 *numbers, taken two and two, are* a, b, c. *What are these numbers?*

Ans. $\frac{a+b-c}{2}$, $\frac{a+c-b}{2}$, $\frac{b+c-a}{2}$.

51. A, B, C, *owe* 2190 *Rs. amongst them, and no one of them can pay this sum alone. But when they unite, it can be done in the following way: by* B's *putting* $\frac{4}{7}$*ths of his property to all* A's; *or by* C's *putting* $\frac{5}{9}$*ths of his property to that of* B; *or by* A's *adding* $\frac{2}{3}$*rds of his property to that of* C. *How much did each possess?*

Ans. A 1530 *Rs.*, B 1540 *Rs.*, C 1170 *Rs.*

52. A *and* B *possess together only* $\frac{2}{3}$*rds of the property of a third* C; B *and* C *have together six times as much as* A; *were* B 680 *Rs. richer than he actually is, then he would have as much as* A *and* C *together. How much has each?*

Ans. A *has* 200 *Rs.*, B 360 *Rs.*, *and* C 840 *Rs.*

53. *I have* 3 *purses lying before me, each of which contains a certain sum of money. If I take* 20 *Rs. out of the* 1*st, and*

*put them into the 2nd, then the latter contains 4 times as much as the former. If, on the other hand, I take 60 **Rs.** out of the 2nd and put them into the 3rd, then this contains $1\frac{3}{4}$ times as much as there is in the 2nd. Again, if I take 40 **Rs.** out of the 3rd, and put them into the 1st, then the 3rd contains $2\frac{7}{8}$ times as much as the 1st. What were the contents of each purse?*

*Ans. In the 1st 120 **Rs.**, in the 2nd 380 **Rs.**, and in the 3rd 500 **Rs.***

54. A, B, C *compare their fortunes.* A *says to* B, *give me 700 Rs. of your money, and I shall have twice as much as you retain;* B *says to* C, *give me 1400 **Rs.** and I shall have thrice as much as you have remaining;* C *says to* A, *give me 420 Rs. and then I shall have five times as much as you retain. How much has each?*

Ans. A 980 ***Rs.***, B 1540 ***Rs.***, C 2380 ***Rs.***

55. *Find 3 numbers of the following properties. If we subtract 4 from the 1st and add as many to the 2nd, then the remainder is to the sum as 1 to 2. If we subtract 10 from the 2nd and add the same number to the 3rd, then the remainder is to the sum as 3 to 10. But if we subtract 5 from the 1st, and add this number to the 3rd, then the remainder is to the sum as 3 to 11. What numbers are they?*

Ans. 20, 28, *and* 50.

56. A, B, C *together possess* 1820 *Rs. If* B *give* A 200 *Rs. of his money, then* A *will have* 160 *Rs. more than* B; *but if* B *receive* 70 ***Rs.*** *from* C, *then both will have the same sum. How much has each?*

Ans. 400 ***Rs.***, 640 ***Rs.***, *and* 780 ***Rs.***

57. *Three persons jointly spent a certain sum; but neither of the 3 is able to pay it of himself.* A, *therefore, says to* B, *give me the 4th part of your money, and then I can pay it alone.* B *says to* C, *give me the 8th part of your money and then I can also pay it; then says* C *to* A, *I shall also be able to pay it if I receive from you the half of your money, although I, at present, only possess 4 **Rs.** How much did they spend, and how much have* A *and* B?

*Ans. They spent 6 **Rs.** 8 ans.;* A *has 5 **Rs.** and* B *6 **Rs.***

58. *A person has 3 pieces of silver of different alloy, viz.* 15, 10, *and* 9 *parts, out of* 16 *pure. If he melt the* 15 *with the* 10, *then there will arise a composition of* $11\frac{2}{3}$ *parts pure. The silver arising from the mixing the* 15 *and* 10 *together will be of the same purity. All three pieces together weigh 34lbs. How much does each piece weigh by itself?*

Ans. The 15 *weighs eight lbs. the* 10 *sixteen, and the* 9 *ten lbs.*

59. *A person has 3 warehouses, each of which contains three kinds of grain, viz. wheat, rye, and barley. The* 1st *warehouse contains* 24 *quarters of wheat,* 9 *quarters of rye and* 15 *quarters of barley; the 2nd,* 9 *quarters of wheat,* 30 *quarters of rye, and* 21 *quarters of barley; the 3rd,* 18 *quarters of wheat,* 27 *quarters of rye, and* 39 *quarters of barley. The value of the* 1st *warehouse is* £734; *the value of the 2nd* £812; *and the value of the 3rd* £1130. *What was the value of a quarter of each grain?*

Ans. The quarter of wheat is worth £56; *the quarter of rye* £42; *and the quarter of barley* £32.

60. A, B, C *purchase coffee, sugar, and tea at the same prices.* A *pays* £11 12*s.* 6*d. for* 7½*lbs. of coffee,* 3 *lbs. of sugar, and* 2¼*lbs. of tea;* B *pays* £16 : 5*s. for* 9*lbs of coffee,* 7 *lbs. of sugar, and* 3 *lbs. of tea;* C *pays* £12 : 5*s. for* 2 *lbs. of coffee,* 5½ *lbs. of sugar, and* 4 *lbs. of tea. What does each cost a lb.?*

Ans. The coffee 15*s.*; *the sugar* 10*s.*; *and the tea* £2.

61. *Three labourers are employed in a certain work.* A *and* B *would together complete this work in* a *days;* A *and* C *require* b *days, but* B *and* C, c *days. What time would each require singly to accomplish it in, supposing, under all circumstances, that each does the same quantity of work? And in what time would they finish it if they all three worked together?*

Ans. A *requires* $\dfrac{2\,abc}{cb + ac - ab}$ *days,*

B $\dfrac{2\,abc}{bc + ab - bc}$ *days,*

and C $\dfrac{2\,abc}{ab + ac - bc}$ *days.*

Jointly they require $\dfrac{2\,abc}{ab + ac + bc}$ *days.*

62. *A person has 3 pieces of metal, each of which consists of gold, silver, and copper. The* 1*st contains* 5 *oz. of gold,* 15 *oz. of silver, and* 30 *oz. of copper; the 2nd contains* 20 *oz. of gold,* 28 *oz. of silver, and* 48 *oz. of copper; the 3rd contains* 12 *oz. of gold,* 39 *oz. of silver, and* 24 *oz. of copper. Now, he wishes to take some from each, and to melt all into a mass, in order to obtain a composition consisting of* 10 *oz. of gold,* 23 *oz. of silver, and* 26 *oz. of copper. How much then must he take of each?*

Ans. Of the 1*st* 10 *oz., of the 2nd* 24 *oz., and of the 3rd* 25 *oz.*

63. *Three soldiers in a battle got* 96 *Rs. booty, which they wish to share equally. In order to do this,* A, *who had obtained most, gives* A *and* B *as much as they already had; in the same manner* B *then divided with* A *and* C, *and after this* C *with* A *and* B. *If, then, by these means, the intended divisions be effected: how much booty did each soldier get?*

Ans. A 52 *Rs.*, B 28 *Rs.*, *and* C 16 *Rs.*

64. *In the* 3 *drawers of my cupboard there are, altogether, a sum of* 162 *Rs. In order that there may be the same sum in all the drawers, I take out of the* 1*st as much as is necessary, and put into each of the other* 2, *the half of what they already contained. I then take out of the* 2*nd and afterwards out of the* 3*rd drawer, and put each time into the other* 2 *drawers the half of what they already contained. If, by these means, I have actually attained my object, how much did each drawer contain at first?*

Ans. In the 1*st* 70 *Rs.*, *in the* 2*nd* 52 *Rs.*, *and in the* 3*rd* 40 *Rs.*

65. A, B, C *play Faro. In the* 1*st game,* A *has the bank,* B *and* C *stake the* 3*rd part of their money and win. In the* 2*nd game,* B *has the bank;* A *and* C *stake the* 3*rd part of their money and also win. Then* C *takes the bank;* A *and* B *stake the* 3*rd part of their money, and this time also the banker loses. After the* 3*rd game they count their money, and find that they all have the same sum, viz.* 64 *Rs. each. How much had they before they began to play?*

Ans. A 75 *Rs.*, B 63 *Rs.*, C 54 *Rs.*

66. A, B, C, D, E *play together on the conditions that he who loses shall give to all the rest as much as they have already. First* A *loses, then* B, *then* C, *then* D, *and at last, also,* E. *All lose in turn, and yet at the end of the* 5*th game they all have the same sum, viz. each* 32 *Rs. How much had each before he began to play?*

Ans. A 81 *Rs.*, B 41 *Rs.*, C 21 *Rs.*, D 11 *Rs.*, *and* E 6 *Rs.*

67. *£*2652 *are divided amongst* 3 *regiments, in such a way, that each man of that regiment which contains most, receives £*1, *and the remainder is divided equally amongst the men of the other* 2 *regiments. Were the pound adjudged to the* 1*st regiment, then each man of the* 2 *remaining regiments would receive* 10*s.*; *if we give the pound to the* 2*nd regiment, then each man of the two remaining regiments would receive* 6*s.* 8*d.*; *lastly, if it were given to the* 3*rd regiment, then each man of the remaining regiments would only receive* 5*s. How strong was each of the* 3 *regiments?*

Ans. The 1*st* 780 *men, the* 2*nd* 1716, *the* 3*rd* 2028.

68. *It is required to determine 3 numbers from the following data. If the 1st be added to m times the others, then the sum = a; if the 2nd be added to the m′ times the others, then the sum = a′; but if the 3rd be added to m″ times the others, then the sum = a″. What numbers are they?*

$$\textit{Making } \frac{m}{m-1} + \frac{m'}{m'-1} + \frac{m''}{m''-1} = A,$$

$$\textit{and } \frac{a}{m-1} + \frac{a'}{m'-1} + \frac{a''}{m''-1} = B;$$

the 3 numbers are, $\frac{1}{m-1}\left(\frac{m\,B}{A-1} - a\right)$; $\frac{1}{m'-1}\left(\frac{m'\,B}{A-1} - a'\right)$,

and $\frac{1}{m''-1}\left(\frac{m''\,B}{A-1} - a''\right)$,

$$\textit{and their sum} = \frac{B}{A-1}.$$

69. *There are 3 numbers whose sum is 83, if 7 be subtracted from the 1st and 2nd, the remainders are to one another as 5 to 3; on the other hand, if three be subtracted from the 2nd and 3rd, then the remainders are to one another as 11 to 9. What numbers are they?*

Ans. 37, 25, 21.

70. *Required to find 3 numbers, which possess the following properties: if 6 be added to the 1st and 2nd, the sums are to one another as 2 to 3; if 5 be added to the 1st and 3rd, then the sums are to one another as 7 to 11; but if 36 be subtracted from the 2nd and 3rd, the remainders are to one another as 6 to 7. What numbers are they?*

Ans. 30, 48, 50.

71. *A certain number consists of 3 digits, which, are in arithmetical progression. If this number be divided by the sum of its digits (consequently without regard to the values which they have by reason of their positions), the quotient is 48; but if we subtract 198 from this number, we obtain a number which has the same digits as the one sought, but in an inverted order. What number is this?*

Ans. 432.

72. *Some smugglers discovered a cave, which would exactly hold the cargo of their boat, viz. 13 bales of cotton, and 33 casks of rum. Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales or casks would it hold?*

Ans. The cave would hold 24 bales or 72 casks.

73. *A purse holds* 19 *crowns and* 6 *guineas. Now* 4 *crowns and* 5 *guineas fill* $\frac{17}{63}$ *of it. How many will it hold of each?*

Ans. 21 *crowns or* 63 *guineas.*

74. *A father, being asked the age of his son, replied: if from double the age he is of now, you subtract three times his age six years ago, you will know what you ask.*

Ans. 9 *years.*

75. *Diophantus, the author of the most ancient book on algebra that has come down to us, passed the sixth part of his life in infancy, the twelfth part of it in adolescency; afterwards he married and lived with his wife the seventh part of his life, and five years more, when he got a son whom he survived four years, and who attained only half the age of his father. At what age did Diophantus die?* *Ans.* 84 *years.*

76. *A trader disposes yearly of Rs* 1000 *for his private expences, still his property increases every year, by a third part of what remained after this deduction, and at the end of three years he finds his capital augmented by a third part of what he had at first. How much had he at the beginning of the* 1st *year?*

Ans. Rs. $3285\frac{5}{7}$.

77. *A trader has two kinds of tea, the one at* 14 *Rs. a seer and the other at* 16 *Rs. How much of each sort must he take, that a box containing* 100 *seers, should be worth Rs.* 1680.

Ans. 30 *seers of the first,*
70 *seers of the second.*

78. *A vase containing* 39 *gallons, was filled in* 12 *minutes; by directing successively two pipes to flow into it, of which one furnished* 4 *gallons per minute the other* 3: *how long did each fountain run?*

Ans. the 1st 3 *minutes, the second* 9.

79. *A watch shewing* 12 *o'clock, the minute hand will cover the hour hand. At what point of the dial will they next be in conjunction?*

Ans. 1 *hour,* 5 *minutes,* 27 *seconds* $\frac{3}{11}$.

N. B. This problem may be compared with that of No. 65, but it may be solved in a much more simple way, by only dividing 12 by 11.

80. *A gentleman meeting some beggars, wishes to give them* 25 *pice each; but finding he wanted* 10 *pice, in order to do so, he gives them only* 20 *pice each; he then found to have* 25 *pice left. It is required to know how much money he had, and how many beggars there were?*

Ans. He had Rs. 2, 9 *ans.* 1 *pice, and the number of beggars* 7.

81. *Three brothers bought a talook for Rs. 50,000; the eldest wanted half the money of the 2nd brother to complete the whole payment; the second could have paid by himself the whole talook, if one-third only of what the eldest had were added to his property: lastly, the youngest son could pay the entire sum with the help of the quarter of the property only of the eldest brother. How much money does each possess?*

Ans. The eldest has Rs. 30,000, *the second Rs.* 40,000, *and the youngest son, Rs.* 42,500.

82. *After the 1st game three gamesters counted their money: one only had lost, the two others gained each as much as he had brought to the play; after a second game, one of the players who had gained before lost, and the two others gained each as much as he had at the beginning of the second game; playing a third time, the player who had gained till now, lost with each of the others as much as each of them had before they began this last game; after which they separated each having Rs.* 120. *How much had they each when they commenced playing?*

The loser at the first game had	*Rs.*	195
He who lost at the second	,,	105
He who lost at the third	,,	60

83. *If a rectangle were* 3 *feet broader and two feet shorter, it would measure* 5 *square feet more, but if it were* 5 *feet broader and only one foot shorter it would measure* 53 *square feet more. What was the measure of the rectangle?*

Ans. 17 *broad and* 15 *feet long.*

84. *A shopkeeper who, in order to weigh any quantity not exceeding one maund or* 40 *seers, had provided himself with* 40 *weights, was shewn that he need not encumber himself with such a number of stones, since the same could be done with four only. What were the weights of these four stones?*

As this question differs much from any of the preceding, we give the method of working it entire.

It is evident that one of the weights must be of 1 seer: let x denote the next greater weight; then in order to weigh 2 seers, $x - 1$ must weigh 2 seers

$$\therefore x - 1 = 2, \text{ or } x = 3.$$

Since $3 - 1 = 2$ and $3 + 1 = 4$; 1, 2, 3 and 4 seers then, may be weighed with two stones, the one weighing 1 and the other 3 seers.

But in order to weigh 5 seers, let y stand for the next greater weight to weigh 5 seers, employing the weights we already have, the value of $y - 3 - 1$ must not exceed 5,

$$\therefore y - 3 - 1 = 5, \text{ or } y = 9.$$

But $9 + 3 + 1 = 13$ consequently any number of seers up to 13 can be weighed with the weights, 1, 3 and 9.

And if u denote the next greater weight; then

$$u - 9 - 3 - 1 = 14 \text{ or } u = 27$$

The law of the series for the least number of weights, is then the geometrical series $1 + 3 + 9 + 27 + 81 + \&c.$

For the present case, the first four terms are sufficient to weigh any whole quantity not above 40 seers.

The next less number of weights is the geometrical series $1 + 2 + 4 + 8 + 16 + 32 + \&c.$; any weight not exceeding 63 can be weighed with weights corresponding to these first 6 terms, and this may be done without placing weights in both scales, or making use of differences as in the former series.

85. *A merchant laid out Rs 3450 in the purchase of three different articles; on the first he gained 20 per cent. and on the 2nd 15 per cent.; the amount gained on each was the same; but on the 3rd article he lost $7\frac{1}{2}$ per cent., which amounted to a sum equal to his gains on the two first articles. How much did he pay for each article?*

$x + y + t = 3450$ 1st equation $\therefore t = 3450 - x - y$

$\frac{20x}{100} = \frac{15y}{100}$ 2nd equation $\therefore y = \frac{20x}{15} = \frac{4}{3}x$

$\frac{20x}{100} + \frac{15y}{100} - \frac{7,5t}{100} = 0$; 3rd equation $\therefore t = \frac{8x + 6y}{3}$

$$\text{Thence } 3450 - x - y = \frac{8x + 6y}{3}$$

$$\text{or } 3450 - x - \frac{4}{3}x = \frac{8x + 8x}{3}$$

$$\text{or } 10350 - 7x = 16x$$

$\therefore x = 450$; and from the 2nd equation

$y = 600$, and from the 1st $t = 3450 - 600 - 450 = 2400$.

86. *A gentleman being asked the age of his three sons, replied, if to the sum of the ages of my three sons, 13 years be added, the result will be four times the age of my youngest son; and if 15 years be added to the difference of the ages of my eldest and youngest sons, it will be the age of my second son; and to get the age of my eldest son, deduct from 83 years the sum of the ages of my three sons. What was the age of each?*

Ans. 16, 17, 18.

87. *A courier sets out from a certain place and travels at the rate of* b *miles in* n *hours; a hours after another courier*

is sent to recal him, he travels at the rate c *miles in* m *hours. After how many hours will the second courier overtake the first?*

Ans $\dfrac{a\,b\,m.}{n\,c - b\,m}$.

88. A *and* B *together can perform a piece of work in* 8 *days,* A *and* C *together in* 9 *days, and* B *and* C *in* 10 *days. In how many days will each perform the same piece of work alone?*

Ans. A *in* $14\frac{34}{49}$ *days,* B *in* $\frac{1723}{41}$ *days, and* C *in* $23\frac{7}{31}$.

As A and B require 8 days to perform the whole work, they will perform in one day the 8th part of the work, viz.

$$A + B = \frac{1}{8} \text{ and for the same reason}$$

$$C + B = \frac{1}{10}$$

$\therefore$ by subtraction $\quad A - C = \dfrac{1}{8} - \dfrac{1}{10} = \dfrac{1}{40}$

again by the question $A + C = \ldots \ldots \ldots = \dfrac{1}{9}$

$\therefore$ by addition $\quad A = \dfrac{49}{720}$; by subtraction $C = \dfrac{31}{720}$

Again, $\quad A + B = \dfrac{1}{8}$

$$A + C = \frac{1}{9}.$$

$\therefore$ by subtraction $\quad B - C = \dfrac{1}{8} - \dfrac{1}{9} = \dfrac{1}{72}$

$$B + C = \ldots \frac{1}{10}$$

$\therefore$ by addition $\quad B = \dfrac{1}{72} + \dfrac{1}{10} = \dfrac{41}{720}$

and for C, we get as found before $\dfrac{31}{720}$

Now if A can finish the $\dfrac{49}{720}$th part of the whole work in one day, in how many days can he finish the whole work?

$\frac{49}{720}$: 1 day :: 1, (the whole work) : $\frac{720}{49} = 14\frac{34}{49}$ days, and so on with the rest.

89. *When a company at a tavern came to pay their reckoning, they found that if there had been 5 more, each would have to pay one and a half rupee less, but if there had been 5 less each would have 3 Rupees more to pay. Required the number of persons, and the quota of each?*

Ans. 15 *persons, quota of each* 6 *Rs.*

90. *A girl being sent to market with two baskets, each containing an equal number of eggs, with the injunction to give from one basket 5 eggs for one ana, but only 3 eggs from the other. To save trouble, she mixed the eggs and sold all of them at the rate of 8 eggs for two anas, but to her great surprise, she brought home one ana less than she should have done had she done as she was bid. What was the number of her eggs?*

Ans. 60.

91. A *and* B *playing at billiards,* A *bet* 5 *Rs. to* 4 *on every game, and found that after a certain number of games he had won* 10 *Rupees. Had* B *won one game more, the number won by him would have been to the number won by* A *as* 3 : 4: *How many games did each win?*

Ans. A *won* 20, *and* B 14.

Of the resolution of any given number of equations of the first degree containing an equal number of unknown quantities.

LXXVIII. When a question has as many distinct conditions as it contains unknown quantities, each of these conditions furnishes an equation, in which it frequently happens, that the unknown quantities are involved with others, as we have already seen in problems with two unknown quantities; but if these unknown quantities are only of the first degree, we may, according to the method adopted in the preceding articles, *find in one of the equations the value of one of the unknown quantities, as if all the rest were known, and substitute this value in all the other equations, which will then contain only the other unknown quantities.*

This operation, by which one of the unknown quantities is exterminated, is called *elimination.* In this way if we have three equations with three unknown quantities, we deduce from them two equations with two unknown quantities, which must be treated as above; and having obtained the values of the

two last unknown quantities, they must be substituted in the expression of the first unknown quantity.

If we have four equations with four unknown quantities, we can deduce, from them, in the first place, three equations with three unknown quantities, which must be treated in the same manner as just described; having then found the values of the three unknown quantities, they must be substituted in the expression for the value of the first unknown quantity, and so on.

The following example, is a question containing three unknown quantities, and from which we can deduce three equations.

LXXIX.—*I bought separately, the contents of three different godowns: the first contained* 30 *measures of rice*, 20 *of gram, and* 10 *of barley, for which I paid Rs* 230.

The second godown, containing 15 *measures of rice*, 6 *of gram, and* 12 *of barley, cost Rs.* 138.

The third godown, contained 10 *measures of rice*, 5 *of gram, and* 4 *of barley, for which I paid Rs.* 75.

It is required to know, how much I must have paid for the measure of rice, how much for that of gram, and for that of barley?

Be x, the price of the measure of rice,
y, that of the measure of gram,
z, that of the measure of barley.

To fulfil the first condition, we observe, that

30	measures of rice	are worth	$30\,x$,
20	ditto gram	are worth	$20\,y$,
10	ditto barley	are worth	$10\,z$;

and as the whole cost of this is Rs. 230, we have the equation

$$30\,x + 20\,y + 10\,z = 230.$$

For the second condition, we have

15	measures of rice,	worth	$15\,x$,
6	ditto gram,	ditto	$6\,y$,
12	ditto barley,	ditto	$12\,z$;

and consequently

$$15\,x + 6\,y + 12\,z = 138.$$

By the third condition, we have

10	measures of rice,	will amount to	$10.x$
5	ditto gram,	..	$5.y$
4	ditto barley,	...	$4.z$

and consequently

$$10\,x + 5\,y + 4\,z = 75$$

The proposed question then will be brought into three equations.

$$30x + 20y + 10z = 230$$
$$15x + 6y + 12z = 138$$
$$10x + 5y + 4z = 75$$

Before proceeding to the resolution of equations, we should examine the equations, to see if it is not possible to simplify them by dividing the two members of some one of them by the same member; we find that, in the present example, the two members of the first are divisible by 10, and those of the second by 3; effecting then those divisions, we get the more simple equations;

$$3x + 2y + z = 23$$
$$5x + 2y + 4z = 46$$
$$10x + 5y + 4z = 75$$

As we may select any of the unknown quantities in order to deduce its value, we give the preference to that of z in the first equation, because this unknown quantity having no co-efficient, its value will be a whole quantity without a divisor, as follows;

$$z = 23 - 3x - 2y.$$

Substituting this value in the second and third equations, they become

$$5x + 2y + 92 - 12x - 8y = 46$$
$$10x + 5y + 92 - 12x - 8y = 75$$

And by reducing the 1st member of each, we find

$$92 - 7x - 6y = 46$$
$$92 - 2x - 3y = 75.$$

These equations contain only two unknown quantities, taking one of these, y, for example from the 1st equation, we obtain

$$y = \frac{92 - 46 - 7x}{6} \text{ or } y = \frac{46 - 7x}{6}$$

and by substituting this value in the 2nd equation, it becomes

$$92 - 2x - 3 \times \frac{46 - 7x}{6} = 75.$$

The denominator 6, might be made to disappear by the usual method, but observing that this denominator 6, is divisible by 3, we may simplify the fraction $\frac{46 - 7x}{6}$, by dividing both its numerator and denominator by 3. We have then

$$92 - 2x - \frac{46 - 7x}{\ } = 75;$$

then multiplying by 2, to cause the denominator to disappear, we get

$$184 - 4x - 46 + 7x = 150,$$

and by reducing the first member, we have

$$138 + 3x = 150,$$

whence we conclude

$$\frac{150 - 138}{3} \quad 12 \quad 4.$$

By the substitution of this value in the expression for that of y, we obtain

$$y = \frac{46 - 7 \times 4}{6} = \frac{46 - 28}{6} = \frac{18}{6} \text{ or } y = 3;$$

and by the substitution of this value of y and that of x in the expression of z, we get

$$z = 23 - 3 \times 4 - 2 \times 3 = 23 - 12 - 6 \text{ or } z = 5.$$

It appears then that a measure of rice was worth	Rs.	4
that of grain,	..	3
that of barley,	..	5.

This example, while it illustrates the method given in the preceding article, ought to be attended to on account of the abbreviations of calculation which are performed in it.

To proceed to another example :—

LXXX. *A sirdar cooly entered into the following conditions with a potter; to carry to town some earthen vessels of three different sizes, forfeiting for each vessel that he broke as much as he received for the transport of those he delivered safe.*

At first he had committed to him two small vessels, four of a middle size, and nine large ones; he broke all the middle sized vessels, and delivered all the rest in good order, and then received the sum of R. 1 : 12, which was the balance in his favor.

The second time there were committed to him seven small vessels, three of the middle size and five large ones; he delivers the small and middle size, but broke the five large ones, when he received only 3 anas.

Lastly, he took charge of nine small vessels, ten middle sized ones, and eleven large ones; breaking these last, he received 4 anas. What was paid for carrying a vase of each size?

Let x be the sum for carrying a small vessel,

y, ... a middle sized one,

z, ... a large one

It is evident, that each sum which the cooly receives, is the difference between what was due to him for the vessels delivered safe, and what he forfeits for those he broke; accordingly the three conditions of the problem furnish respectively the following equations:

$$2x - 4y + 9z = 28$$
$$7x + 3y - 5z = 3$$
$$9x + 10y - 11z = 4.$$

The first of these equations give

$$x = \frac{28 + 4y - 9z}{2}$$

and by substituting this value in the 2nd and 3rd equations, we have

$$7x. \frac{28 + 4y - 9z}{2} + 3y - 5z = 3$$

$$9x. \frac{28 + 4y - 9z}{2} + 10y - 11z = 4.$$

Making the denominators to disappear, we have

$$196 + 28y - 63z + 6y - 10z = 6,$$
$$252 + 36y - 81z + 20y - 22z = 8$$

reducing the first member of each, we obtain

$$196 + 34y - 73z = 6,$$
$$252 + 56y - 103z = 8;$$

taking the value of y in the first of these equations, we find

$$y = \frac{73z - 190}{34}$$

which value of y being substituted, the 2nd equation becomes

$$252 + 56 \times \frac{73z - 190}{34} - 103z = 8,$$

and multiplying each term by 34, the quantity is changed into

$$34 \times 252 + 56 \times 73z - 56 \times 190 - 34 \times 103z = 34 \times 8,$$

or $$8568 + 4088z - 10640 - 3502z = 272;$$

which by the reduction of the 1st member, becomes

$$586z - 2072 = 272;$$

whence we deduce

$$z = \frac{2072 + 272}{586} = \frac{2344}{586} \text{ or}$$

$$z = 4.$$

By going back with the value of z to that of y, we have

$$y = \frac{73 \times 4 - 190}{34} = \frac{292 - 190}{34} = \frac{102}{34} \text{ or}$$

$$y = 3;$$

with these two values of z and y we find

$$x = \frac{28 + 4 \times 3 - 9 \times 4}{2} = \frac{28 + 12 - 36}{2} = \frac{4}{2} \therefore$$

$$x = 2$$

The price, then, of transport for one of the small vessels was 2 ans.
middle sized 3 ,,
large 4 ,,

LXXXI.—It sometimes happens that all the unknown quantities do not enter at the same time into all the equations; the method however is not changed by this circumstance; it is sufficient carefully to examine the connection of the unknown quantities, in order to pass from one to the others.

Let there be, for example, the four following equations:

$$\begin{aligned} 3u - 2y &= 22 \\ 2x + 3y &= 39 \\ 5x - 7z &= 11 \\ 4y + 3z &= 41 \end{aligned}$$

containing the four unknown quantities u, x, y, and z.

It requires but little attention to see that by taking the value of x in the 2nd equation, and substituting it in the 3rd, the results containing only y and z, will, by being combined with the 4th equation, give the values of these two quantities; and having the value of y, we obtain those of x and u, by means of the 1st and 2nd equations.

The following is the process:

$$x = \frac{39 - 3y}{2}, \text{ from the 2nd equation:}$$

which substituted in the 3rd

$$5 \times \frac{39 - 3y}{2} - 7z = 11,$$

$$\text{or } 195 - 15y - 14z = 22,$$

$$\text{or } 15y + 14z = 173,$$

and the 4th equation $\quad 4y + 3z = 41.$

We have from the two last equations

$$y = \frac{173 - 14x}{15} \text{ and from 2nd}$$

$$y = \frac{41 - 3x}{4}$$

$$\therefore \frac{173 - 14x}{15} = \frac{41 - 3x}{4}$$

$$\text{or } 692 - 56x = 615 - 45x$$

$$\text{or } 77 = 11x \therefore x = 7; \therefore y = 5;$$

by means of these values we get

$$x = \frac{39 - 3 \times 5}{2} = \frac{39 - 15}{2} = \frac{24}{2} \text{ or } x = 12,$$

$$\text{and } u = \frac{2 + 2y}{3} = \frac{2 + 2 \times 5}{3} = \frac{12}{3} \text{ or } u = 4:$$

the numbers sought are, consequently,

4, 12, 5 and 7.

LXXXII.—The method we have explained, is applicable to literal as well as to numerical examples; but the multitude of letters which it is necessary to employ to represent generally the data, when the number of equations and of unknown quantities exceeds two, has led algebraists to seek for a more simple manner to express them. Before treating of this process, we must refer the student, desirous of learning to put a problem into an equation and resolving it, to the last series of, questions: at the end of each the answer which ought to be found is given.

General formulæ for the resolution of Equations of the First Degree.

LXXXIII. To obviate the inconvenience which has been remarked at the beginning of the preceding article, we shall represent all the coefficients of the same unknown quantity by the same letter, but distinguish them by one or more accents, according to the number of equations.

General equations with two unknown quantities, are written thus:

$$ax + by = c$$
$$a'x + b'y = c'.$$

The coefficients of the unknown quantity x, are both represented by a, those of y by b; but from the accent which is placed over the letters in the second equation, it can be seen that they are not considered as having the same value, as the corresponding letters in the 1st equation. Thus a' is a quantity different from a; b' is a quantity different from b.

If there are three equations, they are expressed thus:

$$\begin{aligned} ax + by + cz &= d \\ a'x + b'y + c'z &= d' \\ a''x + b''y + c''z &= d''. \end{aligned}$$

All the coefficients of the unknown quantity x, are represented by the letter a, those of y by b, those of z by c; but the several letters are distinguished by a different number of accents, indicating that the quantities so marked have different values. Thus a, a', a'', are three different quantities. The same may be said of the rest.

Agreeably to this method, if we had four unknown quantities and four equations, they should be written thus:

$$\begin{aligned} ax + by + cz + dn &= e \\ a'x + b'y + c'z + d'n &= e' \\ a''x + b''y + c''z + d''n &= e'' \\ a'''x + b'''y + c'''z + d'''n &= e'''. \end{aligned}$$

LXXXIV. To avoid fractions, and simplify the calculation, the process of the diminution may be varied in the following manner.

Be the equations

$$\begin{aligned} ax + by &= c \\ a'x + b'y &= c', \end{aligned}$$

it is evident, that if one of the unknown quantities, x for example, had the same coefficient in the two equations, we have only to subtract one of these equations from the other, in order to make this unknown quantity disappear. This can be seen at once in the equations

$$\begin{aligned} 10\,x + 11\,y &= 27 \\ 10\,x + 9\,y &= 15. \end{aligned}$$

By subtracting the lower from the upper equation, we get

$$11\,y - 9\,y = 27 - 15 \text{ or } 2\,y = 12 \text{ or } y = 6.$$

It is obvious also, that the coefficients of x may be immediately made equal in the equations

$$\begin{aligned} ax + by &= c \\ a'x + b'y &= c'. \end{aligned}$$

By multiplying both members of the 1st equation by a', (the coefficient x in the second,) and both members of the 2nd equation by a, (the coefficient of x in the first;) we obtain

$$a'a\,x + a'by = a'c$$
$$a'a\,x + ab'y = ac'.$$

Then subtracting the first of these equations, from the second, the unknown quantity x will disappear; and we shall only have

$$(ab' - a'b)\,y = ac' - a'c,$$

an equation which contains but one unknown quantity y; whence we deduce

$$y = \frac{ac' - a'c}{ab' - a'b}$$

this proceeding may always be applied to equations of the 1st degree, in order to get rid of any one of the unknown quantities.

Applying this process to three equations containing x, y and z, we can first eliminate x from the 1st and 2nd equations, then from the first and third; we thus arrive at two equations, containing only y and z, from which we may eliminate y for example.

In effecting the calculation the equation containing z to which we shall ultimately arrive, will have a factor common to all its terms, and consequently will not be the most simple term which can be obtained.

LXXXV. A very simple method to eliminate at once all the unknown quantities, except one, has been given by Bézout, by which the question is immediately reduced to equations, which contain one unknown quantity less than those proposed. Although this process is only resorted to when equations of more than two unknown quantities are employed, we shall, in order to give a complete view of this method, begin, by making use of it to equations containing but two unknown quantities.

Be the equations

$$ax + by = c$$
$$a'x + b'y = c';$$

multiplying the first by any indeterminate quantity m, we have

$$amx + bmy = cm;$$

and subtracting from this result the 2nd equation

$$a'x + b'y = c',$$

there remains

$$amx - a'x + bmy - b'y = cm - c'$$

or $$(am - a')\,x + (bm - b')\,y = cm - c'.$$

Since m is an indeterminate quantity, we may suppose it to be such that $bm = b'$; in this case, the term multiplied by y, disappears, and we have

$$x = \frac{cm - c'}{am - a'};$$

but because $bm = b'$, it follows that

$$m = \frac{b'}{b}$$

substituting this value of m into x,

$$x = \frac{c\,\frac{b'}{b} - c'}{a\,\frac{b'}{b} - a'} = \frac{cb' - c'b}{ab' - a'b}$$

If instead of supposing $bm = b'$, we make $am = a'$, the term, which contains x will vanish, and we shall have

$$y = \frac{cm - c'}{bm - b'}.$$

The value of m is no longer the same as in the preceding case, for we have

$$m = \frac{a'}{a}$$

substituting this value into the expression of y, we find

$$y = \frac{ca' - c'a}{ba' - b'a}.$$

LXXXVI. Let there be the three equations

$$\begin{aligned} ax + by + cz &= d \\ a'x + b'y + c'z &= d' \\ a''x + b''y + c''z &= d''. \end{aligned}$$

By an obvious analogy, we shall be led to multiply the first of these equations by m, and the second by n, m and n being indeterminate quantities, to add together the results and to subtract from the sum the third equation; by this means, all the equations will be employed at the same time, and the two new quantities m and n, which we are at liberty to dispose of as we please, will admit of any determinate value which may be necessary to make both the unknown quantities disappear in the result. Having proceeded in this manner, and united the terms by which the same unknown quantity is multiplied, we shall obtain $(am + a'n - a'')x + (bm + b'n - b'')y + (cm + c'n - c'')z = dm + dn' - d''$.

If, it is desired to get rid of x and y, we have only to write the equations

$$am + a'n = a''$$
$$bm + a'n = b''$$

and z is the only unknown quantity remaining;

we now have $$z = \frac{dm + d'n - d''}{cm + c'n - c''}.$$

From the two equations, in which m and n are the unknown quantities, it is easy to deduce the value of these quantities by means of the results obtained in the preceding article; for it is sufficient to change in these results x into m; y into n, and to write instead of the letters

$$\left.\begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix}\right\} \text{ the letters } \left\{\begin{matrix} a, & a', & a'' \\ b, & b', & b'' \end{matrix}\right.$$

by which transformation of letters, we get

$$m = \frac{a''b' - b''a'}{ab' - ba'}.$$
$$n = \frac{ab'' - ba''}{ab' - ba'}*.$$

Substituting these values in that for z, and reducing all the terms to the same denominator, we find

$$z = \frac{d\,(b'a'' - a'b'') + d'(ab'' - ba'') + d''(ab' - ba')}{c(b'a'' - a'b'') + c'(ab'' - ba'') + c''(ab' - ba')}.$$

If the terms containing x and z had been made to disappear, we should have obtained the value of y; the letters m and n would then have depended upon the equations

$$am + a'n = a'' \qquad \text{and} \qquad cm + c'n = c'';$$

and proceeding in the same manner, we should have obtained, for the value of

$$y = \frac{d\,(c'a'' - a'c'') + d'\,(ac'' - ca'') - d''\,(ac' - ca')}{b\,(c'a'' - a'c'') + b\,(ac'' - ca'') - b''\,(ac' - ca')}$$

---

* NOTE.—The value of m and n in the above equations

$$am + a'n = a''$$
$$bm + b'n = b''$$

may also be obtained by the method given before, as

$$abm + a'bn = a''b$$
$$abm + ab'n = ab''$$

$$n\ (ab' - a'b) = ab'' - a''b \text{ or } n = \frac{ab'' - a''b}{ab' - a'b}$$

and likewise multiplying the 1st equation by b', and the 2nd by a', we get the value of m.

Lastly, by assuming the equations

$$bm + b'n = b'' \qquad \text{and} \qquad cm + c'm = c'',$$

we make the terms multiplied by y and by z disappear, and we get for the value of

$$x = \frac{d\,(c'b'' - b'c'') + d'\,(bc'' - cb'') - d''\,(bc' - cb')}{a\,(c'b'' - b'c'') + a'\,(bc'' - cb'') - a''\,(bc' - cb')}$$

Developing these values into such a manner, as to render the terms alternately positive and negative, and changing at the same time, the signs of the numerator and denominator, in the first and third, we shall give them the following forms·

$$\frac{ab'd'' - ad'b'' + da'b'' - ba'd'' + bd'a'' - db'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

$$\frac{ad'c'' - ac'd'' + ca'd'' - da'c'' + dc'a'' - cd'a''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

$$\frac{db'c'' - dc'b'' + cd'b'' - bd'c'' + bc'd'' - cb'd''}{ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a''}$$

LXXXVII. Let there be the four equations

$$\begin{aligned} ax + by + cz + dn &= e \\ a'x + b'y + c'z + d'n &\ \ e' \\ a''x + b''y + c''z + d''n &\ \ e'' \\ a'''x + b'''y + c'''z + d'''n &= e''' \end{aligned}$$

Multiplying the first equation by m, the second bv n, the third by p, and subtracting from the sum of their products, the fourth equation, we shall find

$$\begin{gathered}(am + a'n + a''p - a''')\,x + (bm + b'n + b''p - b''')\,y \\ + (cm + c'n + c''p - c''')\,z + (dm + d'n + d''p - d''')\,u \\ = em + e'n + e''p - e''' \end{gathered}$$

In order to obtain u, we may suppose

$$\begin{aligned} am + a'n + a''p &= a''' \\ bm + b'n + b''p &= b''' \\ cm + c'n + c''p &= c'''. \end{aligned}$$

whence we get

$$u = \frac{em + e'n + e''p - e'''}{dm + d'n + d''p - d'''}$$

The case of four unknown quantities, is now reduced to that of three, by means of the formulæ found for the case of the three unknown quantities which determine the factors m, n, p. It must be observed that the letters d and e do not enter into

these three equations, they neither form any part in the values of m, n, p, and that it follows from thence, *that the numerator of the expression of* u, *is deduced from the denominator, by changing the* d, *which are the coefficients of this unknown quantity, into the corresponding* e, *the known terms in the proposed equations.*

This law, which might have been perceived in the articles 85 and 86, extends to all unknown quantities, whatever be their number. As for their numerator, the observation drawn from the results obtained before, furnishes the means of finding them without any calculation.

LXXXVIII. To reascend to the first link of the chain, let us take the most simple case, the equation with one unknown quantity, $ax = b$; from this we find

$$x = \frac{b}{a},$$

where we see that the numerator is the whole known term b, and the denominator the co-efficient a of the unknown quantity.

From the two equations

$$ax + by = c, \qquad a'x + b'y = c'$$

we have already deduced

$$x = \frac{cb' - bc'}{ab' - ba'}, \qquad y = \frac{ac' - ca'}{ab' - ba'}.$$

The denominator in this case is also composed of the letters a, a', b, b', by which the unknown quantities are multiplied. We first write the letter a by the side of b, which gives ab; we then change the order of a and b and obtain ba; placing the sign — before it, we have $ab - ba$; lastly, we place an accent over the last letter of each term; and the expression becomes $ab' - ba'$ for the denominator.

This done, the numerator of x is obtained, by changing, agreeably to the remark which terminates the preceding article, each a into c; and the numerator of y by changing each b into c; by this means we find for the one is $cb' - bc'$, and for the other $ac' - ca'$.

It requires greater attention to perceive the formation of the denominator in equations with three unknown quantities. However, since in the case of two unknown quantities, the denominator presents all possible arrangements of the two letters a and b, by which these unknown quantities are multiplied, it is natural to conclude, that, when there are three unknown quantities, their denominator must contain all the arrangements of the

three letters a, b, and c. In order to form these arrangements with order, we proceed in the following manner.

We first make the arrangements with the two letters a and b, which are $ab - ba$; then after the first term ab write the third letter c, which gives abc; making this letter pass through all the places, observing each time to change the sign, and not to derange the order in which a and b respectively stand, we obtain

$$abc - acb + cab.$$

Operating in the same manner on the second arrangement of the two letters $- ba$, we shall get

$$- bac + bca - cba;$$

connecting these products with the three preceding, and placing over the second letter one accent, and over the third letter two, we have

$$ab'c'' - ac'b'' + ca'b'' - ba'c'' + bc'a'' - cb'a'',$$

a result, which agrees with that obtained immediately by the formulæ terminating the 86th article.

It is easy to conclude from this, that in order to form a denominator in case of four unknown quantities, we must introduce the letter d in each of the six products,

$$abc - acb + cab - bac + bca - cba;$$

and to make this new letter d, occupy successively all the places, the first term abc, for example, gives the four following;

$$abcd - abdc + adbc - dabc.$$

Proceeding in the same manner with the five other products the total result will be twenty-four terms, in each of which the second letter will have one accent, the third two, and the fourth three.

LXXXIX. That these formulæ may be employed in the resolution of numerical equations, the terms of proposed equations must be compared with the corresponding term of the general equations given in the preceding articles.

To resolve, for example, the three equations

$$\begin{aligned} 7\,x + 5\,y + 2\,z &= 79, \\ 8\,x + 7\,y + 9\,z &= 122, \\ x + 4\,y + 5\,z &= 55, \end{aligned}$$

every term must be compared with the equations given in art. 86. We have then

$$a = 7;\ b = 5;\ c = 2;\ d = 79$$
$$a' = 8;\ b' = 7;\ c' = 9;\ d' = 122$$
$$a'' = 1;\ b'' = 4;\ c'' = 5;\ d'' = 55.$$

Substituting these values in the general expressions for the unknown quantities x, y and z, and going through the operations which are indicated, we find

$$x = 4, \qquad y = 9 \qquad z = 3.$$

It is important to remark, that the same expressions can be employed, even when the proposed equation has not all its terms positive, as it might be supposed from the general equations, from which these expressions are deduced.

For example, let there be,

$$\begin{aligned} 3x - 9y + 8z &= 41 \\ -5x + 4y + 2z &= -20 \\ 11x - 7y - 6z &= 37 \end{aligned}$$

in comparing the terms of these equations with the corresponding terms in the general equations, and having due regard to the signs, we shall have

$$\begin{aligned} a &= +3,\ b = -9,\ c = +8,\ d = +41 \\ a' &= -5,\ b' = +4,\ c' = +2,\ d' = -20 \\ a'' &= +11,\ b' = -7,\ c'' = -6,\ d'' = +37. \end{aligned}$$

We must now determine by the rules of the signs given in art. 31, the sign, which each term of the general expressions of x, y and z ought to have according to the signs of the factors of which it is composed. Thus we find, that the first term of the common denominator, which is $a\,bc''$, becoming $+3 \times +4 \times -6$ changes the sign of the product, and gives -72. Observing the same method with respect to the other terms, as well of the numerators as of the denominators, taking the sum of those terms which are positive, and also that of those that are negative, we obtain

$$x = \frac{2774 - 2834}{592 - 622} = \frac{-60}{-30} = +2$$

$$y = \frac{3022 - 2932}{592 - 622} = \frac{+90}{-30} = -3$$

$$z = \frac{3859 - 3889}{592 - 622} = \frac{-30}{-30} = +1.$$

Equations of the Second Degree, having only one Unknown Quantity.

XC. Hitherto we have been employed upon equations of the *first degree*, or such as involve only the first power of the unknown quantities; but were the question proposed; *To find a number, which, multiplied by five times itself, will give a product equal to* 125; if we designate this number by x, five times this number will be $5x$, and we shall have

$$5x \times x \text{ or } 5x^2 = 125.$$

This is an equation of the *second degree*, because it contains x^2, or the second power of the unknown quantity. If we free this second power from its coefficient 5, we obtain

$$x^2 = \frac{125}{5}, \quad \text{or} \quad x^2 = 25.$$

We cannot here obtain the value of the unknown quantity x, as in art. 11., and the question amounts simply to this, to find a number which, multiplied by itself, will give 25. It is obvious that this number is 5; but it seldom happens that the solution is so easy; hence arises this new numerical question; *to find a number, which, multiplied by itself, will give a product equal to a proposed number*; or, which is the same thing, from the second power of a number, to retrace our steps to the number from which it is derived, and which is called the *square root*. We shall proceed, in the first place, to resolve this question, as it is involved in the determination of the unknown quantities, in all equations of the second degree.

XCI. The method employed in finding or *extracting* the roots of numbers, supposes the second power of such, as are expressed by only one figure, to be known. The following are the nine primitive numbers with their second powers written under them respectively.

1	2	3	4	5	6	7	8	9
1	4	9	16	25	36	49	64	81.

It is evident from this table, that the second power of a number expressed by one figure, contains no more than two figures; 10, which is the least number expressed by two figures, has for its square a number composed of three, 100. In order to resolve the second power of a number consisting of two figures, we must attend to the method by which it is formed; for this purpose we must inquire, how each part of the number 47, for example, is employed in the production of the square of this number.

We may resolve 47 into $40 + 7$, or into 4 tens and 7 units; if

we represent the tens of the proposed number by a, and the units by b, the second power will be expressed by

$$(a + b)(a + b) = a^2 + 2\,a\,b + b^2;$$

that is, it is made up of three parts, namely, *the square of the tens, twice the product of the tens multiplied by the units, and the square of the units.* In the example we have taken, $a = 4$ tens or 40 units, and $b = 7$; we have then

$$\begin{array}{rcr} a^2 & = & 1600 \\ 2\,ab & = & 560 \\ b^2 & = & 49 \\ \hline \end{array}$$

Total, $\quad a^2 + 2\,a\,b + b^2 = 2209 = 47 \times 47.$

Now in order to return, by a reverse process, from the number 2209 to its root, we may observe, that the square of the tens, 1600, has no figure, which denotes a rank inferior to hundreds, and that it is the greatest square, which the 22 hundreds, comprehended in 2209, contains; for 22 lies between 16 and 25, that is, between the square of 4 and that of 5, as 47 falls between 4 tens or 40, and 5 tens or 50.

We find, therefore, upon examination, that the greatest square contained in 22 is 16, the root of which 4 expresses the number of tens in the root of 2209; subtracting 16 hundreds, or 1600, from 2209, the remainder 609 contains double the product of the tens by the units, 560, and the square of the units 49. But as double the product of the tens by the units has no figure inferior to tens, it must be found in the two first figures 60 of the remainder 609, which contain also the tens arising from the square of the units. Now, if we divide 60 by double of the tens 8, and neglect the remainder, we have a quotient 7 equal to the units sought. If we multiply 8 by 7, we have double the product of the tens by the units, 560; substracting this from the whole remainder 609, we obtain a difference 49, which must be, and in fact is, the square of the units.

This process may be exhibited thus:

$$\begin{array}{l|l} 22{,}09 & 47 \\ \hline 16 & 87 \\ \hline 60{,}9 & \\ 60{,}9 & \\ \hline 000 & \end{array}$$

We write the proposed number in the manner of a dividend, and assign for the root the usual place of the divisor. We then separate the units and tens by a comma, and employ only the

two first figures on the left, which contain the square of the tens found in the root. We seek the greatest square 16, contained in these two figures, put the root 4 in its assigned place, and subtract 16 from 22. To the remainder we bring down the two other figures, 09, of the proposed number. separating the last, which does not enter into double the product of the tens by the units, and divide the remainder on the left by 8, double the tens in the root, which gives for the quotient the units 7. In order to collect into one expression the two last parts of the square contained in 609, we write 7 by the side of 8, which gives 87, equal to double the tens plus the units, or $2a + b$; this, multiplied by 7 or b, reproduces $609 = 2ab + b^2$, or double the product of the tens by the units, plus the square of the units. This being subtracted leaves no remainder, and the operation shows, that 47 is the square root of 2209.

If it were required to extract the square root of 324; the operation would be as follows:

$$
\begin{array}{r|l}
3{,}24 & 18 \\
1 & \\
\hline
22{,}4 & 28 \\
22{,}4 & \\
\hline
000 &
\end{array}
$$

Proceeding as in the last example, we obtain 1 for the place of tens of the root; this doubled gives the number 2, by which the two first figures 22 of the remainder are to be divided. Now 22 contains 2 eleven times, but the root can neither be more than 10, nor 10; even 9 is in fact too large, for if we write 9 by the side of 2, and multiply 29 by 9, as the rule requires, the result is 261, which cannot be subtracted from 224. We are, therefore, to consider the division of 22 by 2 only as a means of approximating the units, and it becomes necessary to diminish the quotient obtained, until we arrive at a product, which does not exceed the remainder 224. The number 8 answers to this condition, since $8 \times 28 = 224$; therefore, the root sought is 18.

By resolving the square of 18 into its three parts, we find

$$
\begin{array}{rl}
a^2 &= 100 \\
2ab &= 160 \\
b^2 &= 64 \\
\hline
\text{Total,} \qquad & 324 = 81 \times 18,
\end{array}
$$

and it may be seen, that the 6 tens, contained in the square of the units, being united to 160, double the product of the tens by the units, alters this product in such a manner, that a division of it by double the tens will not give exactly the units.

XCII. It will not be difficult, after what has been said, to extract the square root of a number, consisting of three or four figures; but some further observations, founded upon the principles above laid down, may be necessary to enable the reader to extract the root of any number whatever.

No number less than 100 can have a square consisting of more than four figures, since that of 100 is 10000, or the least number expressed by five figures. In order, therefore, to analyze the square of any number exceeding 100, or 473, for example, we may resolve it into $470 + 3$, or 47 tens plus 3 units. To obtain its square from the formula,

$$a^2 + 2\,ab + b^2,$$

we make $a = 47$ tens $= 470$ units, $b = 3$ units, then

$$\begin{aligned} a^2 &= 220900 \\ 2\,ab &= 2820 \\ b^2 &= 9 \\ \hline \text{Total,}\quad & 223729 = 473 \times 473. \end{aligned}$$

In this example, it is evident that the square of the tens has no figure inferior to hundreds, and this is a general principle, since tens multiplied by tens, always give hundreds.

It is therefore in the part 2237, which remains on the left of the proposed number, after we have separated the tens and units, that it is necessary to seek the square of the tens; and as 473 lies between 47 tens, or 470, and 48 tens, or 480, 2237 must fall between the square of 47 and that of 48; hence the greatest square contained in 2237, will be the square of 47, or that of the tens of the root. In order to find these tens, we must evidently proceed, as if we had to extract the square root of 2237 only; but instead of arriving at an exact result, we have a remainder, which contains the hundreds arising from double the product of the 47 tens multiplied by the units.

The operation is as follows:

$$\begin{array}{l|l}
22,37,29 & 473 \\ \hline
16 & 87 \\ \cline{1-1}
63,7 & 943 \\
60,9 & \\ \hline
282,9 & \\
282,9 & \\ \cline{1-1}
0 &
\end{array}$$

We first separate the two last figures 29, and in order to extract the root of the number 2237, which remains on the left, we further separate the two last figures 37 of this number; the proposed number is then divided into portions of two figures, beginning on the right and advancing to the left. Proceeding with the two first portions as in the preceding article, we find the two first figures 47 of the root; but we have a remainder 28, which, joined to the two figures 29 of the last portion, contains double the product of the 47 tens by the units, and the square of the units. We separate the figure 9, which forms no part of double the product of the tens by the units, and divide 282 by 94, double the 47 tens; writing the quotient 3 by the side of 94, and multiplying 943 by 3, we obtain 2829, a number exactly equal to the last remainder, and the operation is completed.

XCIII. In order to show, by what method we are to proceed with any number of figures, however great, I shall extract the root of 22391824. Whatever this root may be, we may suppose it capable of being resolved into tens and units, as in the preceding examples. As the square of the tens has no figure inferior to hundreds, the two last figures 24 cannot make a part of it; we may therefore separate them, and the question will be reduced to this, to find the greatest square contained in the part 223918, which remains on the left. This part consisting of more than two figures, we may conclude, that the number, which expresses the tens in the root sought, will have more than one figure; it may therefore be resolved, like the others, into tens and units. As the square of the tens does not enter into the two last figures 18 of the number 223918, it must be sought in the figures 2239, which remain on the left; and since 2239 still consists of more than two figures, the square, which is contained in it, must have a root which consists of at least two; the number which expresses the tens sought will therefore have more than one figure; it is then, lastly, in 22 that we must seek the square of that, which represents the units of the highest place in the root required. By this process, which may be extended to any length we please, the proposed number may be divided into portions of two figures from right to left; it must be understood, however, that the last figure on the left may consist of only one figure.

Having divided the proposed number into portions as below, we proceed with the three first portions, as in the preceding article; and when we have found the three first figures 473 of the root, to the remainder 189, we bring down the fourth portion 24, and consider the number 18924, as containing double the

product of the 473 tens already found by the units sought, plus the square of these units. We separate the last figure 4; divide those, which remain on the left, by 946, double of 473, and then make trial of the quotient 2, as in the preceding examples.

22,39,18,24	4732
16	87
63,9	943
60,9	9462
301,8	
282 9	
1892,4	
1892 4	
0000 0	

Here the operation, in the present case, terminates; but it is very obvious, that if we had one portion more, the four figures already found 4732 would express the tens of a root, the units of which would remain to be sought; we should proceed, therefore, to divide the remainder now found, together with the first figure of the following portion, by double of these tens, and so on for each of the portions to be successively brought down.

XCIV. If, after having brought down a portion, the remainder joined to the first figure of this portion, does not contain double of the figures already found, a cipher must be placed in the root; for the root, in this case, will have no units of this rank; the following portion is then to be brought down, and the operation to be continued as before The example subjoined will illustrate this case. The quantities to be subtracted are not put down, but the subtractions are supposed to be performed mentally, as in division.

49,42,0 9	703
04,20,9	1403
0,00,9	

XCV. Every number, it will be perceived, is not a perfect square. If we look at the table given, page 149, we shall see that between the squares of each of the nine primitive numbers, there are intervals comprehending many numbers, which have no assignable root; 45, for instance, is not a square, since it falls between 36 and 49. It very often happens, therefore, that

the number, the root of which is sought, does not admit of one; but if we attempt to find it, we obtain for the result the root of the greatest square, which the number contains. If we seek, for example, the root of 2276, we obtain 47, and have a remainder 67, which shews, that the greatest square contained in 2276, is that of 47, equal to 2209.

As a doubt may sometimes arise, after having obtained the root of a number which is not a perfect square, whether the root found be that of the greatest square contained in the number, we shall give a rule, by which this may be readily determined. As the square of $a + b$ is

$$a^2 + 2\,a\,b + b^2,$$

if we make $b = 1$, the square of $a + 1$ will be

$$a^2 + 2\,a + 1,$$

a quantity which differs from a^2, the square of a, by double of a plus unity. Therefore, *if the root found can be augmented by unity, or more than unity, its square, subtracted from the proposed number, will leave a remainder at least equal to twice this root plus unity.* Whenever this is not the case, the root obtained will be, in fact, that of the greatest square contained in the number proposed.

XCVI. Since a fraction is multiplied by another fraction, when their numerators are multiplied together, and their denominators together, it is evident that the product of a fraction multiplied by itself, or *the square of a fraction is equal to the square of its numerator, divided by the square of its denominator.* Hence it follows, that to extract *the square root of a fraction, we extract the square root of its numerator and that of its denominator.* Thus the root of $\frac{25}{64}$ is $\frac{5}{8}$, because 5 is the square root of 25, and 8 that of 64.

It is very important to remark, that not only are the squares of fractions properly so called, always fractions, but *every fractional number which is irreducible will, when multiplied by itself, give a fractional result, which is also irreducible.*

XCVII. This proposition depends upon the following; *Every prime number* P, *which will divide the product* AB *of two numbers* A *and* B, *will necessarily divide one of these numbers.*

Let us suppose, that it will not divide B, and that B is the greater; if we designate the entire part of the quotient by q, and the remainder by B', we have

$$B = q\,P + B',$$

multiplying by A, we obtain

$$AB = q\,AP + AB',$$

and dividing the members of this equation by P, we have

$$\frac{AB}{P} = q\,A + \frac{AB'}{P};$$

from which it appears, that if AB be divisible by P, the product AB will be divisible by the same number. Now B', being the remainder after the division of B by P, must be less than P; therefore B cannot be divided by P; if we divide P by B' we have a quotient q' and a remainder B''; if further we divide P by B', we have a quotient q'' and a remainder B''', and so on, since P is a prime number.

We have, therefore, the following series of equations;

$$P = q'\,B' + B'',\ P = q''\,B'' + B''', \text{\&c.}$$

multiplying each of these by A, we obtain

$$AP = q'\,AB' + AB'',\ AP = q'\,AB'' + AB''', \text{\&c.}$$

dividing by P, we have

$$A = q'\,\frac{AB'}{P} + \frac{AB''}{P},\qquad A = q''\,\frac{AB''}{P} + \frac{AB'''}{P}, \text{\&c.}$$

From these results it is evident, that if AB' be divisible by P, the products AB'', AB''', &c will also be divisible by it. But the remainders B', B'', B''', &c. are becoming less and less, continually, till they finally terminate in unity, for the operation exhibited above may be continued in the same manner, while the remainder is greater than 1, since P is a prime number. Now when the remainder becomes unity, we have the product $A \times 1$, which must be divisible by P; therefore A also must be divisible by P.

Hence, if the prime number P, which we have supposed not to divide B will not divide A, it will not divide the product of these numbers.

(*This demonstration is taken principally from the* Théorie des Nombres of M. Legendre*.)

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* It is evident that the above proposition may be extended to a product composed of any number of factors, and that if these factors are all of them prime numbers, the product cannot be divided by any other prime number, which shews that the decomposition of a number into simple factors can be effected in one way only.

Moreover, as a number composed of any number of factors, cannot divide another number unless it be successively divisible by each factor of the first, it follows that any number whatever which is prime with one of the factors of a product AB, cannot divide this product unless it divides the other.

XCVIII. Now when the fraction $\frac{b}{a}$ is irreducible, there is no prime number which will divide, at the same time, b and a; but from the preceding demonstration, it is evident, that every prime number, which will not divide a, will not divide $a \times a$, or a^2; every prime number, which will not divide b, will not divide $b \times b$, or b^2, the numbers a^2 and b^2 are, therefore, in this case, prime to each other; and consequently the square $\frac{b^2}{a^2}$ of the fraction $\frac{b}{a}$, being irreducible, as well as the fraction itself, cannot become an entire number.

XCIX. From this last proposition it follows, that *entire numbers, except only such as are perfect squares, admit of no assignable root either among whole numbers or fractions.* Yet it is evident, that there must be a quantity, which, multiplied by itself, will produce any number whatever, 2276, for instance, and that in the present case, this quantity lies between 47 and 48; for 47×47 gives a product less than this number, and 48×48 gives one greater. Dividing then the difference between 47 and 48 by means of fractions, we may obtain numbers, that, multiplied by themselves, will give products greater than the square of 47, but less than that of 48, and which will approach nearer and nearer to the number 2276.

The extraction of the square root, therefore, applied to numbers which are not perfect squares, makes us acquainted with a new species of numbers, in the same manner, as division gives rise to fractions; but there is this difference between fractions and the roots of numbers which are not perfect squares; the former, which are always composed of a certain number of parts of unity, have with unity a *common measure*, or a relation which may be expressed by whole numbers, which the latter have not.

If we conceive unity to be divided into five parts, for example, we express the quotient arising from the division of 9 by 5, or $\frac{9}{5}$, by nine of these parts; $\frac{1}{5}$ then, being contained five times in unity, and nine times in $\frac{9}{5}$, is the *common measure* of unity and the fraction $\frac{9}{5}$, and the relation these quantities have to each other is that of the entire numbers 5 and 9.

This proves the *arithmetical* enunciation, that every fraction of which the terms are prime to each other, is *irreducible*, for let $\frac{b}{a} = \frac{d}{c} \therefore d = \frac{bc}{a}$; but if a and b are prime to each other, and d be a whole or entire number then must a divide c which supposes $c =$ or $> a$ therefore d must also be $=$, or $> b$.

Since whole numbers, as well as fractions, have a common measure with unity, we say that these quantities are *commensurable* with unity, or simply that they are *commensurable;* and since their *relations* or *ratios*, with respect to unity, are expressed by entire numbers, we designate both whole numbers and fractions by the common name of *rational numbers.*

On the contrary, the square root of a number, which is not a perfect square, is *incommensurable* or *irrational*, because, as it cannot be represented by any fraction, into whatever number of parts we suppose unity to be divided, no one of these parts will be sufficiently small to measure exactly, at the same time, both this root and unity.

In order to denote, in general, that a root is to be extracted, whether it can be exactly obtained or not, we employ the character $\sqrt{}$, which is called a *radical* sign;

$\sqrt{16}$ is equivalent to 4,

$\sqrt{2}$ is *incommensurable* or *irrational.*

C. Although we cannot obtain, either among whole numbers or fractions, the exact expression for $\sqrt{2}$, yet we may approximate it, to any degree we please, by converting this number into a fraction, the denominator of which is a perfect square. The root of the greatest square contained in the numerator will then be that of the proposed number expressed in parts, the value of which will be denoted by the root of the denominator.

If we convert, for example, the number 2 into twenty-fifths, we have $\frac{50}{25}$. As the root of 50 is 7, so far as it can be expressed in whole numbers, and the root of 25 exactly 5, we obtain $\frac{7}{5}$, or $1\frac{2}{5}$ for the root of 2, to within one-fifth.

CI. This process, founded upon what was laid down in article 96., that the square of a fraction is expressed by the square of the numerator divided by the square of the denominator, may evidently be applied to any kind of fraction whatever, and more readily to decimals than to others. It is manifest, indeed, from the nature of multiplication, that the square of a number expressed by tenths will be hundredths, and that the square of a number expressed by hundredths will be ten thousandths, and so on; and consequently, that *the number of decimal figures in the square is always double that of the decimal figures in the root.* The truth of this remark is further evident from the rule observed in the multiplication of decimal numbers, which requires that a product should contain as many decimal figures, as there are in both the factors. In any assumed case, therefore, the proposed number, considered as the product of its

root multiplied by itself, must have twice as many decimal figures as its root.

From what has been said, it is clear, that in order to obtain the square root of 227, for example, to within one hundredth, it is necessary to reduce this number to ten thousandths, that is, to annex to it four ciphers, which gives 2270000 ten thousandths. The root of this may be extracted in the same manner, as that of an equal number of units; but to show that the result is hundredths, we separate the two last figures on the right by a comma. We thus find that the root of 227 is 15.06, accurate to hundredths. The operation may be seen below;

$$\begin{array}{r|l} 2,27,00,00 & 1506 \\ \hline 127 & 25 \\ 2\ 00\ 00 & 3006 \\ 19\ 64 & \end{array}$$

If there are decimals already in the proposed number, they should be made even. To extract, for example, the root of 51,7, we place one cipher after this number, which makes it hundredths; we then extract the root of 51,70. If we proposed to have one decimal more, we should place two additional ciphers after this number, which would give 51,7000; we should then obtain 7,19 for the root.

If it were required to find the square root of the number 2 to eleven places of decimals, and that of 3 to fifteen places of decimals, we should annex fourteen ciphers to the first of these numbers, and 30 to the last, the result would be

$$\sqrt{2} = 1,41421356237 \qquad \sqrt{3} = 1,732050807568877.$$

CII. When we have found more than half the number of figures, of which we wish the root to consist, we may obtain the rest simply by division. Let us take, for example, 32976; the square root of this number is 181, and the remainder, 215. If we divide this remainder 215, by 362, double of 181, and extend the quotient to two decimal places, we obtain 0,59, which must be added to 181; the result will be 181,59 for the root of 32976, which is accurate to within one hundredth.

In order to prove that this method is correct, let us designate the proposed number by N, the root of the greatest square contained in this number by a, and that which it is necessary to add to this root to make it the exact root of the proposed number by b; we have then

$$N = a^2 + 2\,ab + b^2,$$

from which we obtain

$$N - a^2 = 2\,ab + b^2;$$

dividing this by $2a$, we find

$$\frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

From this result it is evident, that the first member may be taken for the value of b, so long as the quantity $\frac{b^2}{2a}$ is less than a unit of the lowest place found in b. But as the square of a number cannot contain more than twice as many figures as the number itself, it follows, that if the number of figures in a exceeds double those in b, the quantity $\frac{b^2}{2a}$ will then be a fraction.

In the preceding example, $a = 181$ units, or 18100 hundredths, and consequently contains one figure more than the square of 59 hundredths; the fraction then $\frac{b^2}{2a}$ becomes in this case, $\frac{(59)^2}{2 \times 18100} = \frac{3481}{36200}$, and is less than a unit of the second part 59, or than a hundredth of a unit of the first.

CIII. This leads to a method of approximating the square root of a number by means of vulgar fractions. It is founded on the circumstance, that a, being the root of the greatest square contained in N, b is necessarily a fraction, and $\frac{b^2}{2a}$ being much smaller than b, may be neglected.

If it were required, for example, to extract the square root of 2; as the greatest square contained in this number is 1, if we subtract this, we have a remainder, 1. Dividing this remainder by double of the root, we obtain $\frac{1}{2}$; taking this quotient for the value of the quantity b, we have, for the first approximation to the root, $1 + \frac{1}{2}$ or $\frac{3}{2}$. Raising this root to its square, we find $\frac{9}{4}$, which subtracted from 2 or $\frac{8}{4}$, gives for a remainder $-\frac{1}{4}$. In this case the formula

$$\frac{N - a^2}{2a} = b + \frac{b^2}{2a},$$

becomes

$$-\frac{1}{12} = b + \frac{b^2}{2a}.$$

Substituting $-\frac{1}{12}$ for b, we have for the second approximation $\frac{3}{2} - \frac{1}{12} = \frac{17}{12}$; taking the square of $\frac{17}{12}$, we find $\frac{289}{144}$, a quantity, which still exceeds 2 or $\frac{288}{144}$. Substituting $\frac{17}{12}$ for a, we obtain

$$-\frac{1}{12 \times 34} = b + \frac{b^2}{2a};$$

which gives

$$b = -\frac{1}{12 \times 34} = -\frac{1}{408};$$

the third approximation will then be

$$\frac{17}{12} - \frac{1}{12 \times 34} = \frac{17 \times 34 - 1}{408} = \frac{577}{408}.$$

This operation may be easily continued to any extent we please. We shall give hereafter other formulæ more convenient for extracting roots in general.

CIV. In order to approximate the square root of a fraction, the method, which first presents itself, is, to extract, by approximation, the square root of the numerator and that of the denominator; but with a little attention it will be seen, that we may avoid one of these operations by making the denominator a perfect square. This is done by multiplying the two terms of the proposed fraction by the denominator. If it were required, for example, to extract the square root of $\frac{3}{7}$, we might change this fraction into

$$\frac{3 \times 7}{7 \times 7} = \frac{21}{49}$$

by multiplying its two terms by the denominator, 7. Taking the root of the greatest square contained in the numerator of this fraction, we have $\frac{4}{7}$ for the root of $\frac{3}{7}$, accurate to within $\frac{1}{7}$.

If a greater degree of exactness were required, the fraction $\frac{3}{7}$ must be changed by approximation or otherwise into another, the denominator of which is the square of a greater number than 7. We shall have, for example, the root sought within $\frac{1}{15}$, if we convert $\frac{3}{7}$ into 225ths, since 225 is the square of 15; thus the fraction becomes $\frac{675}{7}$ of one 225th, or $\frac{96}{225}$, within $\frac{1}{225}$; the root of $\frac{96}{225}$ falls between $\frac{9}{15}$ and $\frac{10}{15}$, but approaches nearer to the second fraction than to the first, because 96 approaches nearer to a hundred than to 81; we have then $\frac{10}{15}$ or $\frac{2}{3}$ for the root of $\frac{3}{7}$ within $\frac{1}{15}$.

By employing decimals in approximating the root of the numerator of the fraction $\frac{21}{49}$, we obtain 4,583 for the approximate root of the numerator 21, which is to be divided by the root of the new denominator. The quotient thence arising, carried to three places of decimals, becomes 0,655.

Solution of Equations of the Second Degree.

CV. Before we proceed to a general solution, we shall propose the most simple case, the equation $x^2 = a^2$, which can be put in the form of $x^2 - a^2 = 0$. We have seen, page 35, that $x^2 - a^2$ is the product of $x + a$ by $x - a$, therefore $(x + a)(x - a) = 0$; which may happen in two different ways, either because $x + a = 0$, or because $x - a = 0$; but as there is nothing indicating which of the two values ought to be taken in preference to the other, we say that $x^2 = a^2$ admits of two solutions $x = a$ and $x = -a$: these two solutions are called the roots of the equation. When the second member is not a perfect square, as in $x^2 = a$, the two roots are only indicated, as $x = \sqrt{a}$, and $x = -\sqrt{a}$, or by the expression $x = \pm\sqrt{a}$. Should a have the sign $-$ before it, as in $x^2 = -a$, we then write $x = \pm\sqrt{-a}$.

CVI. We have seen in page 94, that both $+a \times +a$ and $-a \times -a$ give equally the positive product a^2, there can be then no real quantity, which multiplied by itself, can possibly give a product $-a^2$, the expression $\sqrt{-a^2}$ can then only be an imaginary quantity. Thus the equation $x^2 = -a^2$ from which we can only conclude, that $x = \pm\sqrt{-a^2}$, an expression which indicates, that both its roots are imaginary, as no soluble question can lead to such a result. Sometimes however we do arrive at such a result, indicating that the method employed to resolve the problem is not that which ought to have been followed, or that some change in the enunciation ought to be made. From $x^2 = -a^2$, we get $\frac{x^2}{a^2} = -1$, or $\frac{x}{a} = \pm\sqrt{-1}$, or $x = \pm a\sqrt{-1}$. Imaginary quantities are usually represented in this manner. The quantity then of $x^2 + a^2$ must have for its factors, $x + a\sqrt{-1}$, and $x - a\sqrt{-1}$, indeed by actually performing the multiplication

$$\begin{array}{l}
x + a\sqrt{-1} \\
x - a\sqrt{-1} \\
\hline
x^2 + ax\sqrt{-1} \\
\quad - ax\sqrt{-1} - a^2 \times \sqrt{-1} \times \sqrt{-1} \\
\hline
x^2 - a^2 \times \sqrt{-1} \times \sqrt{-1}
\end{array}$$

or, $x^2 - a^2 \times \sqrt{(-1)^2}$, or, $x^2 - a^2 \times (-1)$ or $x^2 + a^2$.

Generally $\sqrt{-m} \times \sqrt{-m}$ is the same as $\sqrt{(-m)^2}$ or $-m$. It might be objected that $(-m)^2$ is the same as m^2, and that m^2 has not $-m$ rather than $+m$ for its root; this is only true when nothing indicates which of the two we ought to chose; but in the present case, the root $-m$ is indicated by the nature of the problem. In fine, the roots $x = \pm\sqrt{-m}$ are imaginary when m is a positive quantity, it may then be represented by

$$x = \pm\sqrt{m}\sqrt{-1}.$$

CVII. An expression as the last, or $a + \sqrt{-m}$, $b - 2\sqrt{-m}$ and generally all those which involve the square root of a negative quantity, are called *imaginary quantities**. They are mere symbols of absurdity that take the place of the value, which we should have obtained, if the question had been possible.

They are not, however to be neglected in the calculation, because it sometimes happens, that when they are combined according to certain laws, the absurdity disappears, and the result becomes real.

CVIII. Let us now resolve the general equation

$$x^2 + px + q = 0.$$

In this equation p and q may represent any quantity positive or negative, fractional or entire. Transposing q, we have $x^2 + px = -q$; if by adding any other indeterminate quantity as m, to each member, the first member were to become the square of some quantity, such as $x + e$, we should have

$$x^2 + px + m = (x + e)^2 = -q + m;$$

whence we get,

$$x + e = \pm\sqrt{-q+m}, \text{ or, } x = -e \pm \sqrt{-q+m}$$

and the problem would be resolved. But as we do not know the value of e, we may observe that the square of $x + e$ is the product of $x + e$ by $x + e$, that is,

$$(x + e)^2 = x^2 + 2ex + e^2$$

which quantity must be identically the same thing as †

* It would be more correct to say, *imaginary expressions*, or *symbols*, as they are not quantities.

† The identity of these two quantities is evident, for as

$$x^2 + px + m = (x + e)^2 = x^2 + 2ex + e^2$$

or the square of any binomial is equal to the square of the 1st term (the square of the 1st term is x^2 in each quantity) plus twice the product of the 1st term by the last, therefore px must be equal to $2ex$, or $p = 2e$, taking away the quantities $x^2 + px = x^2 + 2ex$, there remains only m on the one part, and e^2 on the other, which must consequently be equal.

$$x^2 + p\,x + m.$$

Comparing the two equations, we get $p = 2\,e$, and $e^2 = m$, and consequently

$$e = \frac{p}{2}, \text{ and } m = \frac{p^2}{4};$$

substituting these values of e and m into the equation $x = -e \pm \sqrt{-q+m}$ (preceding page line 6 from below) we get

$$x = -\frac{p}{2} \pm \sqrt{-q + \frac{p^2}{4}}, \text{ or}$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{-4\,q + p^2}{4}} \text{ or } -\frac{p}{2} \pm \tfrac{1}{2}\sqrt{p^2 - 4\,q}.$$

The problem is now resolved, and the solution is real whenever $p^2 - 4\,q$ is a positive quantity, or whenever q being positive but less than $\frac{p^2}{4}$.

If q should be positive but greater than $\frac{p^2}{4}$, the two roots will be imaginary. Expressing $\frac{p^2 - 4\,q}{4}$ by one letter, as $-r$, then

$$x = -\frac{p}{2} \pm \sqrt{r}\,\sqrt{-1}.$$

The two factors of the trinominal $x^2 + p\,x + q$, are then

$x + \frac{p}{2} + \frac{1}{2}\sqrt{p^2 - 4\,q}$ and

$x + \frac{p}{2} - \frac{1}{2}\sqrt{p^2 - 4\,q}$ by effecting the multiplication

$$\begin{array}{l}
x^2 + \frac{p\,x}{2} + \frac{x}{2}\sqrt{p^2 - 4\,q} \\
\qquad \frac{p\,x}{2} + \frac{p^2}{4} + \frac{p}{4}\sqrt{p^2 - 4\,q} \\
\qquad\qquad -\frac{x}{2}\sqrt{p^2 - 4\,q} - \frac{p}{4}\sqrt{p^2 - 4\,q} - \frac{1}{4}(p^2 - 4q) \\
\hline
x^2 + p\,x + \frac{p^2}{4} - \frac{1}{4}(p^2 - 4\,q), \text{ or, } x^2 + p\,x + q
\end{array}$$

the trimonial is thus reproduced. If the two factors are

imaginary, representing one by

$x + \frac{p}{2} + \sqrt{r}\sqrt{-1}$, the other must be

$x + \frac{p}{2} - \sqrt{r}\sqrt{-1}$, and by effecting the multiplication

$$x^2 + \frac{px}{2} + x\sqrt{r}\sqrt{-1}$$
$$+ \frac{px}{2} + \frac{p^2}{4} + \frac{p}{2}\sqrt{r}\sqrt{-1}$$
$$- x\sqrt{r}\sqrt{-1} + \frac{p}{2}\sqrt{r}\sqrt{-1} - r \times (-1)$$

we get

$x^2 + px + \frac{p^2}{4} + r$ and by restoring the value of

$r = -\frac{p^2 - 4q}{4}$, we have $x^2 + px + \frac{p^2}{4} - \frac{p^2 - 4q}{4}$ or

$$x^2 + px + q.$$

CIX. As it is of the greatest importance to acquire just ideas respecting all analytical facts, we shall now give another method to arrive at the same result. We shall begin with equations involving only the second power of the unknown quantity connected with known quantities.

We have only to *collect into one member all the terms containing this power, to free it from the quantities, by which it is multiplied; we then obtain the value of the unknown quantity by extracting the square root of each member.*

Let there be, for example, the equation

$$\tfrac{5}{7}x^2 - 8 = 4 - \tfrac{2}{3}x^2.$$

Making the divisors to disappear, we find first

$$15x^2 - 168 = 84 - 14x^2.$$

Transposing to the first member the term $14x^2$, and to the second the term 168, we have

$$15x^2 + 14x^2 = 84 + 168,$$

or $$29x^2 = 252,$$

and $$x^2 = \tfrac{252}{29},$$

$$x = \sqrt{\tfrac{252}{29}}.$$

It should be carefully observed, that to denote the root of the fraction $\frac{252}{29}$, the sign $\sqrt{}$ is made to descend below the line, which separates the numerator from the denominator. If it were written thus, $\frac{\sqrt{252}}{29}$, the expression would designate the quotient arising from the square root of the number 252 divided by 29; a result different from $\sqrt{\frac{252}{29}}$, which denotes, that the division is to be performed before the root is extracted.

Let there be the literal equation

$$a x^2 + b^3 = c x^2 + d^3 ;$$

proceeding as with the above, we obtain successively

$$a x^2 - c x^2 = d^3 - b^3,$$

$$x^2 = \frac{d^3 - b^3}{a - c},$$

$$x = \sqrt{\frac{d^4 - b^3}{a - c}}.$$

We would remark here, that in order to designate the square root of a compound quantity, the upper line must be extended over the whole radical quantity.

The root of the quantity $4 a^2 b - 2 b^3 + c^3$ is written thus

$$\sqrt{4 a^4 b - 2 b^3 + c^3},$$

or rather

$$\sqrt{(4 a^2 b - 2 b^3 + c^3)},$$

by substituting, for the line extended over the radical quantity, a parenthesis including all the parts of the quantity, the root of which is required. This last expression may often appear preferable to the other.

In general, every equation of the second degree of the kind we are here considering, may, by a transposition of its terms, be reduced to the form

$$\frac{p x^2}{q} = a,$$

$\frac{p}{q}$ designating the coefficient, whatever it may be, of x^2. We then obtain

$$x^2 = \frac{a q}{p},$$

$$x = \sqrt{\frac{a q}{p}}.$$

CX. With respect to numbers taken independently, this solution is complete, since it is reduced to an operation upon the number either entire or fractional, which the quantity $\frac{a\,q}{p}$ represents, an arithmetical operation leading always to an exact result, or to one, which approaches the truth very nearly. But in regard to the signs, with which the quantities may be affected, there remains, after the square root is extracted, an ambiguity, in consequence of which every equation of the second degree admits of two solutions, while those of the first degree admit of only one.

Thus in the general equation $x^2 = 25$, the value of x, being the quantity, which, raised to its square, will produce 25, may, if we consider the quantities algebraically, be affected either with the sign $+$ or $-$; for whether we take $+5$, or -5, for this value we have for the square

$$+5 \times +5 = +25, \text{ or } -5 \times -5 = +25;$$

we may therefore take

$$x = +5,$$

or

$$x = -5.$$

For the same reason, from the general equation

$$x^2 = \frac{a\,q}{p},$$

we have

$$x = +\sqrt{\frac{a\,q}{p}},$$

or

$$x = -\sqrt{\frac{a\,q}{p}}.$$

Both these expressions are comprehended in the following;

$$x = \pm\sqrt{\frac{a\,q}{p}},$$

in which the double sign $\pm$ shews, that the numerical value of

$$\sqrt{\frac{a\,q}{p}},$$

may be affected with the sign $+$ or $-$.

From what has been said, we deduce the general rule, *that the double sign $\pm$ is to be considered as affecting the square root of every quantity whatever.*

It may be here asked, why x, as it is the square root of x^2, is not also affected with the double sign $\pm$? We may answer,

first, that the letter x, having been taken without a sign, that is, with the sign $+$, as the representative of the unknown quantity, it is its value when in this state, which is the subject of inquiry; and that, when we seek a number x, the square of which is b, for example, there can be only two possible solutions; $x = +\sqrt{b}$, $x = -\sqrt{b}$. Again, if in resolving the equation $x^2 = b$, we write $\pm x = \pm \sqrt{b}$, and arrange these expressions in all the different ways, of which they are capable, namely,

$$+x = +\sqrt{b},\ -x = -\sqrt{b},$$
$$+x = -\sqrt{b},\ -x = +\sqrt{b},$$

we come to no new result, since by transposing all the terms of the equations $-x = -\sqrt{b}$, $-x = +\sqrt{b}$, or which is the same thing, by changing all the signs (57), these equations become identical with the first.

CXI. It follows from the nature of the signs, that if the second member of the general equation

$$x^2 = \frac{aq}{p}$$

were a negative number, the equation would be absurd, since the square of a quantity affected either with the sign $+$ or $-$, having always the sign $+$, no quantity, the square of which is negative, can be found either among positive or negative quantities.

This is what is to be understood, when we say, that *the root of a negative quantity is imaginary.*

If we were to meet with the equation

$$x^2 + 25 = 9,$$

we might deduce from it

$$x^2 = 9 - 25,$$

or

$$x^2 = -16;$$

but there is no number, which, multiplied by itself, will produce -16. It is true, that -4 multiplied by $+4$, gives -16; but as these two quantities have different signs, they cannot be considered as equal, and consequently their product is not a square. This species of contradiction, which will be more fully considered hereafter, must be carefully distinguished from that mentioned in art. 58., which disappears by simply changing the sign of the unknown quantity; here it is the sign of the square x^2, which is to be changed.

CXII. To be complete, an equation of the second degree, with only one unknown quantity, must have three kinds of terms, namely, those involving the square of the unknown quantity, others containing the unknown quantity of the first degree, and lastly, such as comprehend only unknown quantities. The following equations are of this kind;

$$x^2 - 4x = 12, \quad 4x - \tfrac{3}{5}x^2 = 4 - 2x.$$

The first is, in some respects, more simple than the second, because it contains only three terms, and the square of x is positive, and has only unity for a coefficient. It is to this last form, that we are always to reduce equations of the second degree, before resolving them; they may then be represented by the general formula,

$$x^2 + px + q = 0,$$

in which p and q denote known quantities, either positive or negative.

It is evident, that we may reduce all equations of the second degree to this state, 1. by collecting into one member all the terms involving x, 2. by changing the sign of each term of the equation, in order to render that of x^2 positive, if it was before negative, 3. by dividing all the terms of the equation by the multiplier of x^2, if this square have a multiplier, or by multiplying by its divisor, if it be divided by any number.

If we apply what has just been said to the equation

$$4x - \tfrac{3}{5}x^2 = 4 - 2x,$$

we have, by collecting into the first member all the terms involving x,

$$-\tfrac{3}{5}x^2 + 6x = 4,$$

by changing the signs,

$$\tfrac{3}{5}x^2 - 6x = -4,$$

multiplying by the divisor 5,

$$3x^2 - 30x = -20,$$

dividing by the multiplier 3,

$$x^2 - 10x = -\tfrac{20}{3}.$$

If we now compare this equation with the general formula

$$x^2 + px + q = 0, \text{ or } x^2 + px = -q,$$

we shall have

$$p = -10, \; -q = -\tfrac{20}{3}.$$

CXIII. In order to arrive at the solution of equations thus prepared, we should keep in mind what has been already

observed p. 35, namely, that the square of a quantity, composed of two terms, always contains the square of the first term, double the product of the first term multiplied by the second, and the square of the second; consequently the first member of the equation

$$x^2 + 2\,a\,x + a^2 = b,$$

in which a and b are known quantities, is a perfect square, arising from $x + a$, and may be expressed thus,

$$(x + a)\;(x + a) = b.$$

If we take the square root of the first member and indicate that of the second, we have

$$x + a = \pm \sqrt{b},$$

an equation, which, considered with respect to x, is only of the first degree; and from which we obtain, by transposition

$$x = -\,a \pm \sqrt{b}.$$

An equation of the second degree may therefore be easily resolved, whenever it can be reduced to the form

$$x^2 + 2\,a\,x + a^2 = b,$$

that is, whenever its first member is a perfect square.

But the first member of the general equation

$$x^2 + p\,x = -\,q$$

contains already two terms, which may be considered as forming part of the square of a binomial; namely, x^2, which is the square of the first term x, and $p\,x$, or double the first multiplied by the second, which second is consequently only half of p, or $\frac{1}{2}\,p$. To complete the square of the binomial $x + \frac{1}{2}\,p$, there must be also the square of the second term, $\frac{1}{2}\,p$; but this square may be formed, since p and $\frac{1}{2}\,p$ are known quantities, and it may be added to the first member, if, to preserve the equality of the two members, it be added at the same time to the second; and this last member will still be a known quantity.

As the square of $\frac{1}{2}\,p$ is $\frac{1}{4}\,p^2$, if we add it to the two members of the proposed equation,

$$x^2 + p\,x = -\,q,$$

we shall have

$$x^2 + p\,x + \tfrac{1}{4}\,p^2 = -\,q + \tfrac{1}{4}\,p^2$$

The first member of this result is the square of $x + \frac{1}{2}\,p$; taking then the root of the two members, we have

$$x + \tfrac{1}{2}\,p = \pm \sqrt{-\,q + \tfrac{1}{4}\,p^2}, \quad (110);$$

by transposition this becomes

$$x = -\tfrac{1}{2}p \pm \sqrt{-q + \tfrac{1}{4}p^2},$$

or which is the same thing

$$x = -\tfrac{1}{2}p + \sqrt{-q + \tfrac{1}{4}p^2},$$

and $$x = -\tfrac{1}{2}p - \sqrt{-q + \tfrac{1}{4}p^2}.$$

We have prefixed the sign — to the second term $\frac{1}{2}p$, of the root of the first member of the above equation, because the second term of this member is positive; the sign — is to be prefixed in the contrary because the square $x^2 - 2ax + a^2$ answers to the binomial $x - a$.

Any equation whatever of the second degree may be resolved by referring it to the general formula,

$$x^2 + px + q = 0;$$

or more expeditiously, by performing immediately upon the equation the operations represented under this formula, which, expressed in general terms, are as follows:

To make the first member of the proposed equation a perfect square, by adding to it, and also to the second, the square of half the given quantity, by which the first power of the unknown quantity is multiplied, then to extract the square root of each member, observing, that the root of the first member is composed of the unknown quantity, and half of the given number, by which the unknown quantity in the second term is multiplied, taken with the sign of this quantity, and that the root of the second member must have the double sign $\pm$, and be indicated by the sign $\sqrt{}$, if it cannot be obtained directly.

See this illustrated by examples.

CXIV. *To find a number such, that if it be multiplied by 7, and this product be added to its square, the sum will be 44.*

The number sought being represented by x, the equation will evidently be

$$x^2 + 7x = 44.$$

In order to resolve this equation, we take $\frac{7}{2}$, half of the coefficient 7, by which x is multiplied; raising it to its square we obtain $\frac{49}{4}$, this added to each member gives

$$x^2 + 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4};$$

reducing the second member to a single term, we have

$$x^2 + 7x + \tfrac{49}{4} = \tfrac{225}{4}$$

The root of the first member, according to the rule given above, is $x + \frac{7}{2}$, and we find for that of the second $\frac{15}{2}$; whence arises the equation

$$x + \tfrac{7}{2} + \pm \tfrac{15}{2},$$

from which we obtain

$$x = -\tfrac{7}{2} \pm \tfrac{15}{2},$$

or

$$x = -\tfrac{7}{2} + \tfrac{15}{2} = \tfrac{8}{2} = 4,$$

$$x = -\tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{22}{2} = -11.$$

The first value of x solves the question in the sense in which it was enunciated, since we have by this value

$$\begin{array}{rl} x^2 = & 16 \\ 7x = & 28 \\ \hline \text{sum} & 44 \end{array}$$

As to the second value of x, since it is affected with the sign —, the term $7x$, which becomes

$$7 \times -11 = -77,$$

must be subtracted from x^2 so that the enunciation of the question resolved by the number -11 is this,

To find a number such, that 7 times this number being subtracted from its square, the remainder will be 44.

The negative value then here modifies the question in a manner, analogous to what takes place, as we have already seen, in equations of the first degree.

If we put the question, as enunciated above, into an equation, we obtain

$$x^2 - 7x = 44,$$

this becomes, when resolved,

$$x^2 - 7x + \tfrac{49}{4} = 44 + \tfrac{49}{4},$$

$$x^2 - 7x + \tfrac{49}{4} = \tfrac{225}{4},$$

$$x - \tfrac{7}{2} = \pm \tfrac{15}{2}.$$

$$x = \tfrac{7}{2} \pm \tfrac{15}{2}$$

$$x = \tfrac{22}{2} = 11,$$

$$x = \tfrac{7}{2} - \tfrac{15}{2} = -\tfrac{8}{2} = -4.$$

The negative value of x becomes positive, as it satisfies precisely the new enunciation, and the positive value, which does not thus satisfy it, becomes negative.

Hence we see, that in equations of the second degree, algebra unites under the same formula two questions, which have a certain analogy to each other.

CXV. Sometimes enunciations, which produce equations of the second degree, admit of two solutions. The following is an example;

To find a number such, that if 15 *be added to its square, the sum will be equal to* 8 *times this number.*

Let x be the number sought; the equation arising from the problem is then

$$x^2 + 15 = 8\,x.$$

This equation reduced to the form prescribed in art. 112 becomes

$$x^2 - 8\,x = -15,$$
$$x^2 - 8\,x + 16 = -15 + 16,$$
$$x^2 - 8\,x + 16 = 1,$$
$$x - 4 = \pm 1,$$
$$x = 4 \pm 1,$$

or

$$x = 5,$$
$$x = 3.$$

There are therefore two different numbers 5 and 3, which fulfil the conditions of the question.

CXVI. Questions sometimes occur, which cannot be resolved precisely in the sense of the enunciation, and which require to be modified. This is the case, when the two roots of the equation are negative, as in the following example,

$$x^2 + 5\,x + 6 = 2.$$

This equation, which denotes, that the *square of the number sought, augmented by* 5 *times this number, and also by* 6, *will give a sum equal to* 2, evidently cannot be verified by addition, as is implied, since 6 already exceeds 2. Indeed, if we resolve it, we find successively

$$x^2 + 5\,x = -4,$$
$$x^2 + 5\,x + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$
$$x + \tfrac{5}{2} = \pm \tfrac{3}{2},$$
$$x = -\tfrac{5}{2} + \tfrac{3}{2} = -1,$$
$$x = -\tfrac{5}{2} - \tfrac{3}{2} = -4.$$

From the sign $-$ with which the numbers 1 and 4 are affected, it may be seen that the term $5\,x$ must be subtracted from the others, and that the true enunciation for both values is,

To find a number such, that if 5 *times this number be subtracted from its square, and* 6 *be added to the remainder, the result will be* 2.

This enunciation furnishes the equation,

$$x^2 - 5\,x + 6 = 2,$$

which gives for x the two positive values 1 and 4.

CXVII. Again, let the following problem be proposed:

To divide a number p *into two parts, the product of which shall be equal to* q.

If we designate one of these parts by x, the other will be expressed by $p - x$, and their product will be $p\,x - x^2$; we have then the equation

$$p\,x - x^2 = q,$$

or, changing the signs,

$$x^2 - p\,x = -q;$$

resolving this last, we find

$$x = \tfrac{1}{2}\,p \pm \sqrt{\tfrac{1}{4}\,p^2 - q}.$$

If now we suppose

$$p = 10, \qquad q = 21,$$

we have

$$x = 5 \pm \sqrt{25 - 21},$$

or

$$x = 5 \pm 2,$$

$$x = 7,$$

$$x = 3,$$

that is, one of the parts will be 7, and the other consequently $10 - 7$, or 3.

If, on the contrary, we take 3 for x, the other part will be $10 - 3$ or 7; so that the enunciation, as it stands, admits, strictly speaking, of only one solution, since the second amounts simply to a change in the order of the parts.

If we examine carefully the value of x in the question we have been considering, we shall see that we cannot take any numbers indifferently for p and q, for if q exceed $\frac{p^2}{4}$ or the square of $\frac{1}{2}\,p$, the quantity $\frac{p^2}{4} - q$, becomes negative, and we are presented with that species of absurdity mentioned in art. 107.

If we take, for example,

$$p = 10 \text{ and } q = 30,$$

we have

$$x = 5 \pm \sqrt{25 - 30} = 5 \pm \sqrt{-5};$$

the problem then, with these assumptions is impossible.

CXVIII. The absurdity of questions, which lead to imaginary roots is discovered only by the result, and we may wish to determine by characters, which are found nearer to the

enunciation, in what consists the absurdity of the problem, which gives rise to that of the solution; this we shall be enabled to do by the following consideration.

Let d be the difference of the two parts of the proposed number; the greater part will be $\frac{p}{2}+\frac{d}{2}$, the less $\frac{p}{2}-\frac{d}{2}$ (7); but it has been proved (29, 30, & 34) that

$$\left(\frac{p}{2}+\frac{d}{2}\right)\left(\frac{p}{2}-\frac{d}{2}\right)=\frac{p^2}{4}-\frac{d^2}{4};$$

therefore, the product of the two parts of the proposed number, whatever they may be, will always be less than $\frac{p^2}{4}$, or than the square of half their sum, so long as d is any thing but zero; when d is nothing, each of the two parts being equal to $\frac{p}{2}$, their product will be only $\frac{p^2}{4}$. It is then absurd to require it to be greater; and it is just, that algebra should answer in a manner contradictory to established principles, and thereby shew, that what is sought does not exist.

What has been proved concerning the equation

$$x^2-px=-q,$$

furnished by the preceding question, is true of all those of the second degree, where q is negative in the second member, the only equations, which produce imaginary roots, since the term $\frac{p^2}{4}$ placed under the radical sign, preserves always the sign $+$, whatever may be that of p. Indeed, it is evident that the equation

$$x^2+px=-q, \text{ or } x^2+px+q=0,$$

will admit of no positive solution, since the first member contains only affirmative terms; and, to ascertain whether the unknown quantity x can be negative, we have only to change x into $-y$. The unknown quantity y would then have positive values, which would be furnished by the equation

$$y^2-py+q=0, \text{ or } y^2-py=-q,$$

which is precisely the same as that in the preceding article; but as the values of x can be real only when those of y would be so, they become therefore imaginary in the case under consideration, when q exceeds $\frac{p^2}{4}$.

It will be perceived then from what has been said, how and for what reason, *when the known term of an equation of the second degree is negative in the second member, and greater than the square of half the coefficient of the first power of the unknown quantity, this equation can have only imaginary roots.*

CXIX. The expressions

$$\sqrt{-b},\ a+\sqrt{-b},$$

and, in general, those, which involve the square root of a negative quantity, are called *imaginary quantities.* They are mere symbols of absurdity, that take the place of the value, which we should have obtained, if the question had been possible.

CXX. We shall shew that, *if there exists a quantity* a, *which substituted in the place of* x, *verifies the equation of the second degree,* $x^2 + p\,x = -q$, *and is consequently the value of* x, *this unknown quantity will still have another value.* Now, if we substitute a for x, the result will be $a^2 + p\,a = -q$; and since, by supposition, a represents the value of x, $-q$ will be necessarily equal to the quantity $a^2 + p\,a$; we may then write this quantity in the place of $-q$, in the proposed equation, which thus becomes

$$x^2 + p\,x = a^2 + p\,a.$$

Transposing all the terms of the second member, we have

$$x^2 + p\,x - a^2 - p\,a = 0,$$

which may be written,

$$x^2 - a^2 + p\,(x - a) = 0;$$

and because

$$x^2 - a^2 = (x + a)\,(x - a.)\quad (34),$$

it is obvious, at once, that the first member is divisible by $x - a$, and will give an exact quotient, namely, $x + a + p$; we have then,

$$x^2 + p\,x - q = x^2 - a^2 + p\,(x - a) = (x - a)\,(x + a + p).$$

Now it is evident, that a product is equal to zero, when any one of its factors whatever becomes nothing; we shall have then

$$(x - a)\,(x + a + p) = 0,$$

not only when $x - a = 0$, which gives

$$x = a,$$

but also when $x + a + p = 0$, from which is deduced

$$x = -a - p.$$

Therefore, if a is one of the values of x, $-a-p$ will necessarily be the other.

This result agrees with the two values comprehended in the formula

$$x = -\tfrac{1}{2}p \pm \sqrt{-q + \tfrac{1}{4}p^2};$$

for if we take for a the first value, $-\frac{1}{2}p + \sqrt{-q + \frac{1}{4}p^2}$, we obtain for the other

$$-a-p = +\tfrac{1}{2}p - \sqrt{-q+\tfrac{1}{4}p^2} - p = -\tfrac{1}{2}p - \sqrt{-q+\tfrac{1}{4}p^2},$$

which is in fact the second value.

These remarks contain the germ of the general theory of equations of whatever degree, as will appear hereafter, when the subject will be resumed.

CXXI. The difficulty of putting a problem into an equation, is the same in questions involving the second and higher powers as in those involving only the first, and consists always in disentangling and expressing distinctly in algebraic characters all the conditions comprehended in the enunciation. The preceding questions present no difficulty of this sort; and, although the learner is supposed to be well exercised in those of the first degree, I shall proceed to resolve a few questions, which will furnish occasion for some instructive remarks.

A person employed two laborers, allowing them different wages; the first received, at the end of a certain number of days, 96 francs, and the second, having worked six days less, received only 54 francs; if this last had worked the whole number of days, and the other had lost six days, they would both have received the same sum; it is required to find how many days each worked, and what sum each received for a day's work.

This problem, which at first view appears to contain several unknown quantities, may be easily solved by means of one, because the others may be readily expressed by this.

If x represent the number of days' work of the first laborer, $x - 6$ will be the number of days' work of the second,

$\dfrac{96^{\text{fr.}}}{x}$ will be the daily wages of the first,

$\dfrac{54}{x-6}$ the daily wages of the second;

if this last had worked x days, he would have earned

$$\times \frac{54}{x-6} \text{ or } \frac{54x}{x-6},$$

and the first working $x - 6$ days, would have received only

$$(x-6)\quad 96 \quad \text{or} \quad 96\,(x-6)$$

The equation of the problem then will be

$$\frac{54\,x}{x-6} = \frac{96\,(x-6)}{x}$$

The first step is to make the denominators disappear; the equation then becomes

$$54\,x^2 = 96\,(x-6)\,(x-6).$$

As the numbers 54 and 96 are both divisible by 6, the result may be simplified by division; we shall then have

$$9\,x^2 = 16\,(x-6)\,(x-6).$$

This last equation, may be prepared for solution according to the rule given in art. 112, but as the object of this rule is to enable us with more facility to extract the root of each member of the equation proposed, it is here unnecessary, because the two members are already presented under the form of squares; for it is evident, that $9\,x^2$ is the square of $3\,x$, and $16\,(x-6)\,(x-6)$ the square of $4\,(x-6)$. We have then

$$3\,x = \pm\, 4\,(x-6);$$

from which may be deduced

$$3\,x = 4\,x - 24,\ x = 24,$$
$$3\,x = -\,4\,x + 24,\ x = \tfrac{24}{7}.$$

By the first solution, the first laborer worked 24 days, and consequently earned $\frac{96}{24}$ or 4 francs per day, while the second worked only 18 days, and received $\frac{54}{18}$ or 3 francs per day.

The second solution answers to another numerical question, connected with the equation under consideration, in a manner analogous to what was noticed in art. 115.

CXXII. *A banker receives two notes against the same person; the first of 550 francs, payable in seven months, the second of 720 francs, payable in four months, and gives for both the sum of 1200 francs; it is required to find what is the annual rate of interest, according to which these notes are discounted.*

In order to avoid fractions in expressing the interest for seven months and four months, we shall represent by $12\,x$ the interest of 100 francs for one year; the interest for one month will then be x. The present value of the first note will accordingly be found by the proportion,

$$100 + 7x : 100 :: 550 : \frac{55000}{100 + 7x}$$

and the present value of the second note by the proportion,

$$100 + 4x : 100 :: 720 : \frac{72000}{100 + 4x}.$$

By uniting these values, we obtain for the equation of the problem,

$$\frac{55000}{100 + 7x} + \frac{72000}{100 + 4x} = 1200.$$

Dividing each of the members by 200, we have

$$\frac{275}{100 + 7x} + \frac{360}{100 + 4x} = 6;$$

making the denominators disappear, we find successively,

$275(100 + 4x) + 360(100 + 7x) = 6(100 + 7x)(100 + 4x)$,
$27500 + 1100x + 36000 + 2520x = 60000 + 6600x + 168x^2$,
which may be reduced to

$$168x^2 + 2980x = 3500;$$

dividing by 2, we obtain

$$84x^2 + 1490x = 1750,$$

which gives

$$x^2 + \frac{1490}{84}x = \frac{1750}{84}.$$

Comparing this equation with the formula,

$$x^2 + px = -q,$$

we have

$$p = \frac{1490}{84}, \quad -q = \frac{1750}{84}$$

and the expression

$$x = -\frac{1}{2}p \pm \sqrt{\frac{p^2}{4} - q},$$

becomes

$$x = -\frac{745}{84} \pm \sqrt{\frac{745 \cdot 745}{84 \cdot 84} + \frac{1750}{84}}.$$

Reducing the fractions, we have

$$\frac{745\,.\,745 + 1750\,.\,84}{84\,.\,84} = \frac{702025}{84\,.\,84};$$

then, since the denominator of this fraction is a perfect square, we have only to extract the square root of its numerator. If we stop at thousandths, we find 837,869, for the root of 702025; this, taken with the denominator 84, gives for the values of x

$$x = -\frac{745}{84} + \frac{837,869}{84} \qquad \frac{92.869}{84},$$

$$-\frac{745}{84} - \frac{837,869}{84} \qquad - \frac{1582,869}{84}.$$

The first of these values is the only one, which solves the question in the sense, in which it was enunciated. Dividing the denominator of this fraction by 12, we have

$$12\,x = \frac{92,869}{7} = 13,267;$$

that is, the annual interest is at the rate of 13,27 nearly.

CXXIII. The following question deserves attention on account of the character, which the expression for the unknown quantity presents.

To divide a number into two parts, the squares of which shall be in a given ratio.

Let a be the given number,
 m the ratio of the squares of its two parts,
 x one of these parts;
the other will be $a - x$.
We shall then have, according to the enunciation,

$$(a - x)\ (a - x) \qquad m.$$

This may be resolved in two ways; we may either reduce it to the form $x^2 + p\,x = q$, and then resolve it by the common method; or since the fraction

$$\overline{(a - x)\ (a - x)}$$

is a square, the numerator and denominator being each a square, we thence conclude at once,

$$\frac{x}{a-x} = \pm\sqrt{m},$$

$$x = \pm(a-x)\sqrt{m}.$$

By resolving separately the two equations of the first degree comprehended in this formula, namely,

$$x = +(a-x)\sqrt{m},$$

$$x = -(a-x)\sqrt{m},$$

we have

$$x = \frac{a\sqrt{m}}{1+\sqrt{m}},$$

$$x = \frac{-a\sqrt{m}}{1-\sqrt{m}}.$$

By the first solution, the second part of the number proposed is

$$a - \frac{a\sqrt{m}}{1+\sqrt{m}} = \frac{a + a\sqrt{m} - a\sqrt{m}}{1+\sqrt{m}} = \frac{a}{1+\sqrt{m}};$$

and the two parts,

$$\frac{a\sqrt{m}}{1+\sqrt{m}} \text{ and } \frac{a}{1+\sqrt{m}}$$

are both, as the enunciation requires, less than the number proposed.

By the second solution we have

$$a + \frac{a\sqrt{m}}{1-\sqrt{m}} = \frac{a - a\sqrt{m} + a\sqrt{m}}{1-\sqrt{m}} = \frac{a}{1-\sqrt{m}};$$

and the two parts are

$$-\frac{a\sqrt{m}}{1-\sqrt{m}} \text{ and } \frac{a}{1-\sqrt{m}}.$$

Their signs being opposite, the number a is strictly no longer their sum, but their difference.

If we make $m = 1$, that is, if we suppose that the squares of the two parts sought are equal, we have

$$\sqrt{m} = 1;$$

and the first solution will give two equal parts,

$$\frac{a}{2}, \quad \frac{a}{2};$$

a conclusion, that is self-evident, while the second solution gives for the results two infinite quantities (68), namely,

$$\frac{-a}{1-1} \text{ or } \frac{-a}{0}, \text{ and } \frac{a}{1-1} \text{ or } \frac{a}{0}.$$

This is necessary, for it is only by considering two quantities infinitely great, with respect to their difference a, that we can suppose the ratio of their squares equal to unity.

Now, let there be the two quantities, x, and $x - a$, the ratio of their squares will be

$$\frac{x^2}{x^2 - 2\,ax + a^2};$$

dividing the two terms of this fraction by x^2, we obtain

$$\frac{1}{1 - \frac{2\,a}{x} + \frac{a^2}{x^2}};$$

but it is evident, that the greater the number x, the less will be the fractions $\frac{2\,a}{x}$, $\frac{a^2}{x^2}$, and the more nearly will the above ratio approach to $\frac{1}{1}$, or 1.

CXXIV. Now in order to compare the general method with that, which we have just employed, we develope the equation

$$\frac{x^2}{(a-x)\,(a-x)} = m,$$

and we have, successively,

$$x^2 = m\ (a-x)\ (a-x),$$
$$x^2 = a^2 m - 2\,a\,m\,x + m\,x^2,$$
$$x^2 - m\,x^2 + 2\,a\,m\,x = a^2\,m,$$
$$(1-m)\,x^2 + 2\,a\,m\,x = a^2\,m,$$
$$x^2 + \frac{2\,a\,m\,x}{1-m} = \frac{a^2 m}{1-m},$$

making $$p = \frac{2\,a\,m}{1-m}, \quad -q = \frac{a^2 m}{1-m},$$

the general formula gives,

$$x = -\frac{a\,m}{1-m} \pm \sqrt{\frac{a^2\,m^2}{(1-m)\,(1-m)} + \frac{a^2\,m}{1-m}}.$$

These values of x appear very different from those, which were found above; yet they may be reduced to the same; and in this consists the utility of the example, on which we are employed. It will serve to show the importance of those transformations, which different algebraic operations produce in the expression of quantities.

We must first reduce the two fractions comprehended under the radical sign to a common denominator. This may be done by multiplying the two terms of the second by $1-m$; we have then

$$\frac{a^2\,m^2}{(1-m)\,(1-m)} + \frac{a^2\,m}{1-m} = \frac{a^2\,m^2 + a^2\,m\,(1-m)}{(1-m)\,(1-m)}$$

$$\frac{a^2\,m^2 + a^2\,m - a^2\,m^2}{(1-m)\,(1-m)} = \frac{a^2\,m}{(1-m)\,(1-m)}.$$

The denominator being a square, it is only necessary to extract the root of the numerator; we then have

$$\sqrt{\frac{a^2\,m^2}{(1-m)\,(1-m)} + \frac{a^2\,m}{1-m}} = \frac{\sqrt{a^2\,m}}{1-m};$$

but the expression $\sqrt{a^2\,m}$ may be further simplified.

It is evident that the square of a product is composed of the product of the squares of each of its factors, for example,

$$b\,c\,d \times b\,c\,d = b^2,\, c^2,\, d^2,$$

and consequently the root of $b^2\,c^2\,d^2$ is simply the product of the roots b, c, and d, of the factors b^2, c^2, and d^2. Applying this principle to the product $a^2\,m$, we see that its root is the product of a, the root of a^2, by $\sqrt{m}$, which denotes the root of n, or that

$$\sqrt{a^2\,m} = a\,\sqrt{m}.$$

It follows from these different transformations, that

$$x = -\frac{a\,m}{1-m} \pm \frac{a\,\sqrt{m}}{1-m}.$$

or $$\frac{am - a\sqrt{m}}{1-m},$$

$$x = -\frac{am + a\sqrt{m}}{1-m}.$$

These expressions, however simple, are still not the same as those given in the preceding article; if, moreover, we seek to verify them for the case, in which $m = 1$, they become

$$x = \frac{-a + a}{1-1} = \frac{0}{0},$$

$$x = \frac{-a - a}{1-1} = \frac{-2a}{0}.$$

We find, in the second, the symbol of infinity, as in the preceding article, but the first presents this indeterminate form, $\frac{0}{0}$, of which we have already seen examples in articles 69 and 70; and before we pronounce upon its value, it is proper to examine, whether it does not belong to the case stated in art. 70: whether there is not some factor common to the numerator and denominator, which the supposition of $m = 1$ renders equal to zero.

The expression $$\frac{-am + a\sqrt{m}}{1-m}$$

may be resolved into

$$\frac{a(-m + \sqrt{m})}{1-m} = \frac{a(\sqrt{m} - m)}{1-m}.$$

It is here evident, that the numerator does not become 0, except by means of the factor $\sqrt{m} - m$; we must therefore examine, whether this last has not some factor in common with the denominator $1 - m$. In order to avoid the inconvenience arising from the use of the radical sign, let us make $\sqrt{m} = n$, then taking the squares, we have $m = n^2$; the quantities, therefore,

$$\sqrt{m} - m \text{ and } 1 - m$$

become $$n - n^2 \text{ and } 1 - n^2,$$

but $n - n^2 = n(1 - n)$, and $1 - n^2 = (1 - n)(1 + n)$ (84); restoring to the place of n its value $\sqrt{m}$, we have

$$\sqrt{m} - m = (1 - \sqrt{m})\sqrt{m},$$
$$1 - m = (1 - \sqrt{m})(1 + \sqrt{m})$$

and consequently,

$$\frac{a(\sqrt{m} - m)}{1 - m} = \frac{a(1 - \sqrt{m})\sqrt{m}}{(1 - \sqrt{m})(1 + \sqrt{m})} = \frac{a\sqrt{m}}{1 + \sqrt{m}},$$

a result the same, as that found in art. 119.

In the same manner we may reduce the second value of x, observing that

$$\frac{-a\sqrt{m} - a\,m}{1 - m} = \frac{-a(1 + \sqrt{m})\sqrt{m}}{(1 - \sqrt{m})(1 + \sqrt{m})} = \frac{-a\sqrt{m}}{1 - \sqrt{m}},$$

as in art. 119*.

It will be seen without difficulty, that we might have avoided radical expressions in the preceding calculations, by taking m^2 to represent the ratio, which the squares of the two parts of the proposed number have to each other; m would then have been the square root, which may always be considered as known, when the square is known; but we could not have perceived from the beginning the object of such a change in a given term, of which algebraists often avail themselves, in order to render calculations more simple. It is recommended to the learner, therefore, to go over the solution again, putting m^2 in the place of m.

Examples.

1. Given $x^2 + 4x = 140$, to find the value of x.

Here $x^2 + 4x = 140$, by the question,

Whence $x = -2 \pm \sqrt{4 + 140}$, by the rule,

Or, which is the same thing, $x = -2 \pm \sqrt{144}$,

Therefore $x = -2 + 12 = 10$, or $-2 - 12 = -14$,

Where one of the values of x is positive and the other negative.

* The example, which we have given at some length, corresponds with a problem resolved by Clairaut, in his Algebra, the enunciation of which is as follows: *To find on the line, which joins any two luminous bodies, the point where these two bodies shine with equal light.* We have divested this problem of the physical circumstances, which are foreign to the object of this work, and which only divert the attention from the character of the algebraic expressions. These expressions are very remarkable in themselves, and for this reason we have developed them more fully, than they were done in the work referred to.

2. Given $x^2 - 12x + 30 = 3$, to find the value of x.
Here $x^2 - 12x = 3 - 30 = -27$, by transposition,
Whence $x = 6 \pm \sqrt{36 - 27}$, by the rule,
Or, which is the same thing, $x = 6 \pm \sqrt{9}$,
Therefore $x = 6 + 3 = 9$, or $= 6 - 3 = 3$,
Where it appears that x has two positive values.

3. Given $2x^2 + 8x - 20 = 70$, to find the value of x.
Here $2x^2 + 8x = 70 + 20 = 90$, by transposition,
And $x^2 + 4x = 45$, by dividing by 2,
Whence $x = -2 \pm \sqrt{4 + 45}$, by the rule.
Or, which is the same thing, $x = -2 \pm \sqrt{49}$,
Therefore $-2 + 7 = 5$, or $= -2 - 7 = -9$,
One of the values of x being positive and the other negative.

4. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$, to find the value of x.

Here $3x^2 - 3x = 5\frac{1}{3} - 6 = -\frac{2}{3}$ by transposition,

And $x^2 - x = -\frac{2}{9}$ by dividing by 3,

Whence $x = \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - \frac{2}{9}\right)}$, by the rule,

Or, by subtracting $\frac{2}{9}$ from $\frac{1}{4}$, $x = \frac{1}{2} \pm \sqrt{\frac{1}{36}}$,

Therefore $x = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, or $= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$,

In which case x has two positive values.

5. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 20\frac{1}{2} = 42\frac{2}{3}$, to find the value of x.

Here $\frac{1}{2}x^2 - \frac{1}{3}x = 42\frac{2}{3} - 20\frac{1}{2} = 22\frac{1}{6}$ by transposition,

And $x^2 - \frac{2}{3}x = 44\frac{1}{3}$, by dividing by $\frac{1}{2}$, or mult. by 2,

Whence we have $x = \frac{1}{3} \pm \sqrt{\left(\frac{1}{9} + 44\frac{1}{3}\right)}$, by the rule,

Or, by adding $\frac{1}{9}$ and $44\frac{1}{3}$ together, $x = \frac{1}{3} \pm \sqrt{\frac{400}{9}}$,

Therefore $x = \frac{1}{3} + 6\frac{2}{3} = 7$, or $= \frac{1}{3} - 6\frac{2}{3} = -6\frac{1}{3}$,

Where one value of x is positive, and the other negative.

6. Given $ax^2 + bx = c$, to find the value of x.

Here $x^2 + \frac{b}{a}x = \frac{c}{a}$ by dividing each side by a.

Whence, by the rule, $x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b^2}{4a^2} + \frac{c}{a}\right)}$.

Or, multiplying c and a by $4a$, $x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 + 4ac}{4a^2}}$,

Therefore $x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 + 4ac}$.

7. Given $ax^2 - bx + c = d$, to find the value of x.

Here $ax^2 - bx = d - c$, by transposition,

And $x^2 - \frac{b}{a}x = \frac{d-c}{a}$, by dividing by a.

Whence $x = \frac{b}{2a} \pm \sqrt{\left(\frac{d-c}{a} + \frac{b^2}{4a}\right)}$ by the rule,

Or, multg. $d - c$ & a by $4a$, $x = \frac{b}{2a} \pm \frac{1}{2a}\sqrt{4a(d-c) + b^2}$.

8. Given $x^4 + ax^2 = b$, to find the value of x.

Here $x^4 + ax^2 = b$, by the question,

Or $x^2 = -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)} = -\frac{a}{2} \pm \frac{1}{2}\sqrt{(a^2 + 4b)}$, by the rule,

Whence $x = \pm\sqrt{\left(-\frac{a}{2} \pm \frac{1}{2}\sqrt{4b + a^2}\right)}$ by extraction of roots.

9. Given $\frac{1}{2}x^6 - \frac{1}{4}x^3 = -\frac{1}{32}$, to find the value of x.

Here $\frac{1}{2}x^6 - \frac{1}{4}x^3 = -\frac{1}{32}$, by the question,

And $x^6 - \frac{1}{2}x^3 = -\frac{1}{16}$, by multiplying by 2,

Whence $x^3 = \frac{1}{4} \pm \sqrt{\left(\frac{1}{16} - \frac{1}{16}\right)} = \frac{1}{4}$, by the rule,

And consequently $x = \sqrt[3]{\frac{1}{4}} = \sqrt[3]{\frac{2}{8}} = \frac{1}{2}\sqrt[3]{2}$.

10. Given $2x\frac{2}{3} + 3x\frac{1}{3} = 2$, to find the value of x.

Here $2x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 2$, by the question,

And $x^{\frac{2}{3}} + \frac{3}{2}x^{\frac{1}{3}} = 1$, by dividing by 2,

Whence $x^{\frac{1}{3}} = -\frac{3}{4} \pm \sqrt{\left(\frac{9}{16} + 1\right)} = -\frac{3}{4} \pm \frac{5}{4} = \frac{1}{2}$, or -2,

Therefore $x = (\frac{1}{2})^3 = \frac{1}{8}$, or $(-2)^3 = -8$.

11. Given $x^4 - 12x^3 + 44x^2 - 48x = 9009\ (a)$, to find the value of x.

This equation may be expressed as follows,

$$(x^2 - 6x)^2 + 8(x^2 - 6x) - a,$$

Whence $x^2 - 6x = -4 \pm \sqrt{16 + a}$, by the common rule,

And, by a second operation, $x = 3 \pm \sqrt{(9 - 4 \pm \sqrt{16 + a})}$

Therefore, by restoring the value of a, we have

$$x = 3 \pm \sqrt{5 \pm \sqrt{9025}},$$

Or, by extraction of roots, $x = 13$, the Ans.

12. Given

$(x + y)(x^2 + y^2) = 580$ or $x^3 + x^2y + xy^2 + y^3 = 580$

$(x - y)(x^2 - y^2) = 160$ or $x^3 - x^2y - xy^2 + y^3 = 160$

$$2x^2y + 2xy^2 = 420$$

$$\begin{aligned} x^3 + x^2y + xy^2 + y^3 &= 580 \\ 2x^2y + 2xy^2 &= 420 \\ \hline x^3 + 3x^2y + 3xy^2 + y^3 &= 1000, \text{ or } x + y = 10\ (A) \end{aligned}$$

$(x - y)(x^2 - y^2) = (x - y)(x - y)(x + y) = 160$, or

$$10(x - y)^2 = 160, \therefore (x - y)^2 = \frac{160}{10} = 16 \therefore x - y = 4\ (B)$$

$$\begin{aligned} (A)\quad x + y &= 10 \\ (B)\quad x - y &= 4 \\ \hline \text{by addition,}\quad 2x &= 14 \\ x &= 7 \\ \text{by subtraction,}\quad 2y &= 6 \\ y &= 3 \end{aligned}$$

Examples for Practice.*

1. Given $x^2 - 8x + 10 = 19$, to find the value of x.
Ans. $x = 9$.

2. Given $x^2 - x - 40 = 170$, to find the value of x.
Ans. $x = 15$.

3. Given $3x^2 + 2x - 9 = 76$, to find the value of x.
Ans. $x = 5$.

4. Given $\frac{1}{2}x^2 - \frac{1}{3}x + 7\frac{5}{9} = 8$, to find the value of x.
Ans. $x = 1\frac{1}{3}$.

5. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$, to find the value of x.
Ans. $x = 49$.

6. Given $x + \sqrt{5x + 10} = 8$, to find the value of x.
Ans. $x = 3$.

7. Given $\sqrt{10 + x} - \sqrt[4]{10 + x} = 2$, to find the value of x.
Ans. $x = 6$.

8. Given $2x^4 - x^2 + 96 = 99$, to find the value of x.
Ans. $x = \frac{1}{2}\sqrt{6}$.

9. Given $x^6 + 20x^3 - 10 = 59$, to find the value of x.
Ans. $x = \sqrt[3]{3}$.

10. Given $3x^{2n} - 2x^n + 3 = 11$, to find the value of x.
Ans. $x = \sqrt[n]{2}$.

11. Given $\frac{2}{3}x\sqrt{3 + 2x^2} = \frac{1}{2} + \frac{2}{3}x^2$, to find the value of x.
Ans. $x = \frac{1}{2}\sqrt{-3 + 3\sqrt{2}}$.

12. Given $x\sqrt{\left(\frac{6}{x} - x\right)} = \frac{1 + x^2}{\sqrt{x}}$, to find the value of x.
Ans. $x = (1 + \frac{1}{2}\sqrt{2})^{\frac{1}{2}}$.

13. Given $\frac{1}{x}\sqrt{1 - x^3} = x^2$, to find the value of x.
Ans. $x = \left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)^{\frac{1}{3}}$.

* The unknown quantity in these examples, as well as in those given above, has always two values, as appears from the rule; but the negative roots, being, in general, but seldom used in practical questions of this kind, are here suppressed as far as example 20.

14. Given $x\sqrt{\left(\frac{a}{x}-1\right)}=\sqrt{x^2-b^2}$, to find the value of x.

Ans. $x=\frac{1}{4}a+\frac{1}{4}\sqrt{8b^2+a^2}$.

15. Given $\sqrt{1+x-x^2}-2(1+x-x^2)=\frac{1}{9}$, to find the value of x.

Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$.

16. Given $\sqrt{\left(x-\frac{1}{x}\right)}+\sqrt{\left(1-\frac{1}{x}\right)}=x$, to find the value of x.

Ans. $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

17. Given $x^{4n}-2x^{3n}+x^n=6$, to find the value of x.

Ans. $x=\sqrt[n]{\frac{1}{2}+\frac{1}{2}\sqrt{13}}$

18. $\sqrt{4x+\frac{x^2}{3}}-\sqrt{x}=x+3\sqrt{x}+\frac{4}{3}x$.

Ans. $x=9$

19. $ax^n-bx^{\frac{n}{2}}-c=-d$

$$x=\left(\frac{b\pm\sqrt{4ac-4ad+b^2}}{2a}\right)^{\frac{2}{n}}$$

20. $ax^2=b$

$x=+\sqrt{\frac{b}{a}}$, $x=-\sqrt{\frac{b}{a}}$

21. $x^2+6x=27$

$x=3$, $x=-9$

22. $x^2-7x+3\frac{1}{4}=0$

$x=6\frac{1}{2}$, $x=\frac{1}{2}$

23. $x^2-5\frac{3}{4}x=18$

$x=8$, $x=-2\frac{1}{4}$

24. $3x^2-2x=65$

$x=5$, $x=-4\frac{1}{3}$

25. $622x=15x^2+6384$

$x=22\frac{4}{5}$, $x=18\frac{2}{3}$

26. $20748-1616x+21x^2=0$

$x=60\frac{2}{3}$, $x=16\frac{2}{7}$

27. $9\frac{3}{5}\,x - 21\frac{15}{16} = x^2$

$x = 5\frac{17}{20},\ x = 3\frac{3}{4}$

28. $11\frac{3}{4}\,x - 3\frac{1}{2}\,x^2 = -41\frac{1}{4}$

$x = -2\frac{1}{7},\ x = 5\frac{1}{2}$

29. $9\frac{1}{3}\,x^2 - 90\frac{1}{3}\,x + 195 = 0$

$x = 6\frac{3}{7},\ x = 3\frac{1}{4}$

30. $18\,x^2 + \frac{18078}{65}\,x + 4728 = 0$

$x = -25\frac{10}{39},\ x = \quad 52$

31. $x^2 - 8\,x = 14$

$x = 4 + \sqrt{30},\ x = 4 - \sqrt{30}$

Or $x = 9{,}4772\ldots,\ x = -1{,}4772\ldots$

32. $3\,x^2 + x = 7$

$x = \frac{-1 + \sqrt{85}}{6},\ x = \frac{-1 - \sqrt{85}}{6}$

Or $x = 1\cdot3699\ldots,\ x = -1\cdot7032$

33. $118\,x - 2\frac{1}{2}\,x^2 = 20$

$x = \frac{118 + \sqrt{13724}}{5},\ x = \frac{1\cdot18 - \sqrt{13724}}{5}$

Or $x = 47\cdot0298\ldots,\ x = 0\cdot1701\ldots$

34. $6\,x - 30 = 3\,x^2$

$x = 1 + \sqrt{-9},\ x = 1 - \sqrt{-9}$

35. $8\,x^2 - 7\,x + 34 = 0$

$x = \frac{7 + \sqrt{-1039}}{16},\ x = \frac{7 - \sqrt{-1039}}{16}$

36. $4\,x^2 - 9\,x = 5\,x^2 - 255\frac{3}{4} - 8\,x$

$x = 15\frac{1}{2},\ x = -16\frac{1}{2}$

37. $80\,x + \frac{3\,x^2}{4} + \frac{21\,x - 27782}{12} = 1859\frac{1}{3} - 3x^2$

$x = -46,\ x = 24\frac{1}{3}$

38. $\frac{x}{x + 60} = \frac{7}{3\,x - 5}$

$x = 14,\ x = -10$

39. $\frac{40}{x - 5} + \frac{27}{x} = 13$

$x = 9,\ x = 1\frac{2}{13}$

40. $\frac{8x}{x+2} - 6 = \frac{20}{3x}$

$x = 10,\ x = -\frac{2}{3}$

41. $\frac{48}{x+3} = \frac{165}{x+10} - 5$

$x = 5\frac{2}{5},\ x = 5$

42. $\frac{31}{6x} = \frac{16}{117 - 2x} + 1$

$x = 67\frac{1}{6},\ x = 4\frac{1}{2}$

43. $\frac{2x+3}{10-x} = \frac{2x}{25 - 3x} - 6\frac{1}{2}$

$x = 13\frac{22}{31},\ x = 8$

44. $\frac{25x + 180}{10x - 81} = \frac{40x}{5x - 8} - \frac{3}{5}$

$\cdot\frac{3}{5}\ x = \frac{72}{245}$

45. $\frac{18 + x}{6(3 - x)} = \frac{20x + 9}{19 - 7x} - \frac{65}{4(3 - x)}$

$x = 7\frac{22}{115},\ x = 2\frac{1}{2}$

46. $a d x - a c x^2 = b c x - bd$

$x = \frac{d}{c},\ x = -\frac{b}{a}$

47. $\frac{a^2 x^2}{f^4} - \frac{2ax}{g} + \frac{f^2}{g^2} = 0$

$x = \frac{f^2}{ag},\ x = \frac{f^2}{ag}$

48. $c x + \frac{ac}{a+b} = (a+b) x^2$

$x = \frac{c + \surd(c^2 + 4ac)}{2(a+b)},\ x = \frac{c - \surd(c^2 + 4ac)}{2(a+b)}$

49. $a x^2 + b^2 + c^2 = a^2 + 2bc + 2(b - c) x \surd a$

$x = \frac{b - c + a}{\surd a},\ x = \frac{b - c - a}{\surd a}$

50. $c x^2 - 2cx \surd d = d x^2 - cd$

$x = \frac{\surd cd}{\surd c + \surd d},\ x = \frac{\surd cd}{\surd c - \surd d}$

51. $3 \surd(112 - 8x) = 19 + \surd(3x + 7)$

$x = 6$

52. $\frac{x}{3} - \frac{y}{15} = 1$

$x^2 - 4\left(\frac{y}{9}\right)^2 \quad 44$

$x = 12$, and $19\frac{11}{19}$

$y = 45$, and $82\frac{17}{19}$

53 $3x^2 - \frac{xy}{3} + y^2 = 361$

$x^2 - y^2 + \frac{2}{3}xy = 134$

$x = 11 \qquad y = 3$

54. $\frac{x^2}{9} - y^2 - \frac{2}{5}x = 0$

$3x - 18y^2 - \frac{x}{5} = 0$

$x = 5 \qquad y = \frac{\sqrt{7}}{3}$

55 $\frac{10x + y}{xy} = 3$

$9y - 9x = 18$

$x = 2$, or $-\frac{1}{3}$; $\qquad y = 4$, or

56 $\sqrt{x^2 - 40} + \sqrt[4]{x - 40}$

$2x^2 - y^2 + x = -3$

$x = 11 \qquad y = 16$

57. $x - y = 15$

$\frac{xy}{2} = y^3$

$x = 18$ or $\frac{25}{2}$; $y =$

58. $\frac{10x + y}{xy} = 3$

$9y - 9x = 18$

2 or $-\frac{1}{3}$; $y =$ [illegible] or $\frac{5}{2}$

2 c

59. $x + y : x - y :: 13 : 5$

$y^2 + x = 25$

$x = 9$, or $-\frac{225}{16}$; $y = 4$, or $-\frac{25}{4}$

60. $4xy = 96 - x^2y^2$

$x + y = 6$

$x = 4$, or 2, or $3 \pm \sqrt{21}$

$y = 2$, or 4, or $3 \mp \sqrt{21}$

61. $x^n + y^n = 2a^n$

$xy = b^2$

$$x = \left(a^n \pm \sqrt{a^{2n} - b^{2n}}\right)^{\frac{1}{n}}$$

$$y = \frac{b^2}{x} = \frac{b^2}{(a^n \pm \sqrt{a^{2n} - b^{2n}})^{\frac{1}{n}}}$$

62. $x^2 + x + y = 18 - y^2$

$xy = 6$

$x = 3$, or 2, or $-3 \pm \sqrt{3}$

$y = 2$, or 3, or $-3 \pm \sqrt{3}$

63. $x^2 + 2xy + y^2 + 2x = 120 - 2y$

$xy - y^2 = 8$

$x = 6$, or 9, or $-9 \mp \sqrt{5}$

$y = 4$, or 1, or $-3 \pm \sqrt{5}$

64. $x^2 + y^2 - x - y = 78$

$xy + x + y = 39$

$x = 9$, or 3, or $\frac{-13 \pm \sqrt{-39}}{2}$

$y = 3$, or 9, or $\frac{-13 \mp \sqrt{-39}}{2}$

65. $x^2y^4 - 7xy^2 - 945 = 765$

$xy - y = 12$

$x = 5$, or $\frac{1}{5}$, or $\frac{-19}{17 \mp 6\sqrt{-2}}$; $y = 3$, or -15, or, $-6 \pm \sqrt{-2}$

66. $x - 2\sqrt{xy} + y - \sqrt{x} + \sqrt{y} = 0$

$\sqrt{x} + \sqrt{y} = 5$

$x = 9$, or $\frac{25}{4}$

$y = 4$, or $\frac{25}{4}$

67. $\frac{x^2}{y^2} + \frac{4x}{y} = \frac{85}{9}$

$x - y = 2$

$x = 5, \text{ or } \frac{17}{10}; \; y = 3, \text{ or } \frac{-3}{10}$

68. $\sqrt{\frac{3x}{x+y}} + \sqrt{\frac{x+y}{3x}} = 2$

$xy - x - y = 54$

$x = 6, \text{ or } -\frac{9}{2}; \; y = 12, \text{ or } -9$

69. $x^4 - 2x^2y + y^2 = 49$

$x^4 - 2x^2y^2 + y^4 - x^2 + y^2 = 20$

$x = \pm\sqrt{6}, \text{ or } \pm\sqrt{\frac{-13 \pm \sqrt{-47}}{2}},$

$\text{or } \pm\sqrt{\frac{15 \pm 3\sqrt{5}}{2}} \quad \text{or } \pm\sqrt{\frac{-13 \pm \sqrt{-11}}{2}}$

$y = 2 \text{ or } -1 \text{ or } \frac{1 \pm \sqrt{-47}}{2}, \text{ or } \frac{1 \pm 3\sqrt{5}}{2}$

$\text{or } \frac{1 \pm \sqrt{-11}}{2}$

70. $xy + xy^2 = 12$

$x + xy^3 = 18$

$x = 2, \text{ or } 16; \; y = 2, \text{ or } \frac{1}{2}$

71. $x - x^{\frac{1}{2}} = 3 - y$

$4 - x = y - y^{\frac{1}{2}}$

$x = 4 \text{ or } \frac{1}{4}; \quad y = 1, \text{ or } \frac{9}{4}$

72. $(x^2 + 1)\,y = x\,y + 126$

$(x^2 + 1)\,y = x^2y^2 - 744$

$x = 5, \text{ or } \frac{1}{5}, \text{ or } \frac{-97 \pm \sqrt{6045}}{58}$

$y = 6, \text{ or } 150, \text{ or } \frac{1682}{97 \mp \sqrt{6045}}$

73. $x^2 + \frac{7}{44}x = \frac{1695}{12716}$

$x = \frac{5}{17}$

74. $x^2 - \frac{21}{37}x = -\frac{3960}{49247}$

$$x = \frac{3}{11}$$

75. $x^2 - \frac{4}{19}x = \frac{39559}{93347}$

$$x = \frac{13}{17}$$

76. $\sqrt{a + x} - b\,(a + x)^{\frac{1}{4}} = m$

$$x = \left(\frac{b}{2} \pm \sqrt{m + \frac{b^2}{4}}\right)^4 - a$$

77. $x\,y = a$

$x^2 + y^2 = b$

$$x = \pm\sqrt{\frac{b \pm \sqrt{(b^2 - 4a^2)}}{2}}$$

$$y = \pm\sqrt{\frac{b \mp \sqrt{(b^2 - 4a^2)}}{2}}$$

78. $x + y = a$

$x^3 + y^3 = b$

$$x = \frac{a}{2} \pm \sqrt{\frac{4b - a^3}{12a}}$$

$$y = \frac{a}{2} \mp \sqrt{\frac{4b - a^3}{12a}}$$

79. $2x + 3y = 118$

$5x^2 - 7y^2 = 4333$

$$x = 35, \text{ or } -229\tfrac{6}{17}$$

$$y = 16, \text{ or } \quad 192\tfrac{4}{17}$$

80. $x^2 - y^2 = b$

$(x + y + a)^2 + (x - y + a)^2 = c$

$$x = \pm\frac{-a \pm \sqrt{(2b + c - a^2)}}{2}$$

$$y = \pm\frac{[illegible] - b \mp a\sqrt{(2b + c - a^2)}}{2}$$

81. $\frac{18\,x}{y} = \frac{8\,y}{x}$

$3\,x\,y + 2\,x + y = 485$

$x = 10, \text{ or } -10\frac{7}{9}, \text{ or } \frac{1 \pm \sqrt{-34919}}{18}$

$y = 15, \text{ or } -16\frac{1}{6}, \text{ or } \frac{-1 \mp \sqrt{-34919}}{12}$

82. $\frac{a\,x}{y} = \frac{b\,y}{x}$

$c\,x\,y + d\,x + my = n$

$$x = \frac{\pm(m\sqrt{a} \pm d\sqrt{b}) \pm \sqrt{[(m\sqrt{a} \pm d\sqrt{b})^2 \pm 4\,c\,n\sqrt{a\,b}]}}{2\,c\sqrt{a}}$$

$$y = \mp\frac{(m\sqrt{a} \pm d\sqrt{b}) \pm \sqrt{[(m\sqrt{a} \pm d\sqrt{b})^2 \pm 4\,c\,n\sqrt{a\,b}]}}{2\,c\sqrt{b}}$$

83. $x^2 + x + y^2 + y = m$

$x^2 + x - y^2 - y = n$

$x = \frac{-1 \pm \sqrt{(2\,m + 2\,n + 1)}}{2}$

$y = \frac{-1 \pm \sqrt{(2\,m - 2\,n + 1)}}{2}$

84. $x + y = x\,y$

$x + y + x^2 + y^2 = a$

$$x = \frac{1 \pm \sqrt{(4\,a + 1)} + \sqrt{[4\,a - b \mp b\sqrt{(4\,a + 1)}]}}{4}$$

$$y = \frac{1 \pm \sqrt{(4\,a + 1)} - \sqrt{[4\,a - b \mp b\sqrt{(4a + 1)}]}}{4}\,^*$$

85. $a\,x - b\,y = m$

$a^3\,x^3 - b^3\,y^3 = n\,x\,y$

$x = \frac{m}{a}\left(1 \pm \sqrt{\frac{n + a\,b\,m}{n - 3\,a\,b\,m}}\right)$

$y = \frac{m}{2\,b}\left(-1 \pm \sqrt{\frac{n + a\,b\,m}{n - 3\,a\,b\,m}}\right)$

86. $x^2 + y^2 + z^2 = a$

$y^2 = 2\,x\,z + b$

$c\,x = d\,z$

$x = \frac{d\sqrt{(a - b)}}{c + d}\,;\quad y = \frac{\sqrt{[2\,a\,c\,d + b\,(c^2 + d^2)]}}{c + d}$

$z = \frac{c\sqrt{(a - b)}}{c + d}$

* These values of x and y may also be interchanged.

87. $x(y+z)=a$

$y(x+z)=b$

$z(x+y)=c$

$$x=\pm\sqrt{\frac{(a-c+b)(a-b+c)}{2(b+c-a)}}$$

$$y=\pm\sqrt{\frac{(a-c+b)(b-a+c)}{2(a+c-b)}}$$

$$z=\pm\sqrt{\frac{(a-b+c)(c-a+b)}{2(a+b-c)}}$$

88. $\dfrac{x\,y\,z}{x+y}=a$

$\dfrac{x\,y\,z}{y+z}=b$

$\dfrac{x\,y\,z}{x+z}=c$

$$x=\pm\sqrt{\frac{2\,a\,b\,c\,(a\,b+b\,c-a\,c)}{(a\,b+a\,c-b\,c)\,(a\,c+b\,c-a\,b)}}$$

$$y=\pm\sqrt{\frac{2\,a\,b\,c\,(a\,c+b\,c-a\,b)}{(a\,b+a\,c-b\,c)\,(a\,b+b\,c-a\,c)}}$$

$$z=\pm\sqrt{\frac{2\,a\,b\,c\,(a\,b+a\,c-b\,c)}{(a\,b+b\,c-a\,c)\,(a\,c+b\,c-a\,b)}}$$

89. $x\,y=p$

$(b-y)\,z=p'$

$(a-x)\,(c-z)=p''$

$$x=\frac{-A\pm\sqrt{[A^2-4\,p\,(p'-b\,c)\,(p''-a\,c)]}}{2\,(p'-b\,c)}$$

$$y=\frac{-A\mp\sqrt{[A^2-4\,p\,(p'-b\,c)\,(p''-a\,c)]}}{2\,(p''-a\,c)}$$

$$z=\frac{-B\mp\sqrt{[B^2-4\,p'\,(p-b\,c)\,(p''-a\,c)]}}{2\,(p-a\,b)}$$

$A=c\,p-a\,p'-b\,p''+a\,b\,c$

$B=c\,p-a\,p'+b\,p''-a\,b\,c$

90. $5-2\sqrt{y+2}=\dfrac{9x^2}{64}-(\sqrt{x}-3\sqrt{y})^2$

$\dfrac{7}{y}-10\sqrt{\dfrac{x}{y}}=x-16$

$x=4;\quad y=\dfrac{1}{4}$

91. $x^4 + y^4 = 1 + 2\,x\,y + 3\,x^2y^2$

$x^3 + y^3 = 1 + x + 2\,y^2\,x + 2\,y^2$

$x = 2$, or -1; $y = 1$

92. $\dfrac{x + y + \sqrt{x^2 - y^2}}{x + y - \sqrt{x^2 - y^2}} = \dfrac{9}{8\,y}\,(x + y)$

$(x^2 + y^2) + x - y = 2\,x\,(x^2 + y^2) + 506$

$x = 5$, or $-\dfrac{23}{5}$, or $\dfrac{1 \pm \sqrt{-\text{[illegible]}}}{5}$

$y = 3$, or $-\dfrac{69}{25}$, or $\dfrac{3 \pm 3\sqrt{-1209}}{25}$

93. $\dfrac{x + \sqrt{x + y}}{x - \sqrt{x + y}} - \dfrac{\sqrt{x - x - y}}{\sqrt{x - x + y}} = \dfrac{9}{20}$

$9\,y^2 - 9\sqrt{x\,y^2} = 4\,x$

$x = 9$, or $\dfrac{196}{9}$, or $\dfrac{289}{9}$ or, 16

$y = 4$, or $\dfrac{-14}{9}$, or $\dfrac{-68}{9}$, or $\dfrac{4}{3}$

94. $x + y - \sqrt{\dfrac{x + y}{x - y}} = \dfrac{6}{x - y}$

$x^2 + y^2 = 41$

$x = \pm 5$, or $\pm 3\sqrt{\dfrac{5}{2}}$

$y = \pm 4$, or $\pm 4\sqrt{\dfrac{37}{2}}$

95. $\dfrac{x^4}{y^2} + \dfrac{y^4}{x^2} = 136\dfrac{1}{9} - 2\,x\,y$

$x + 4 = 14 - y$

$x = 6$, or 4, or $5 \pm 5\sqrt{-\frac{13}{11}}$

$y = 4$, or 6, or $5 \mp 5\sqrt{-\frac{13}{11}}$

96. $\sqrt{5\sqrt{x} + 5\sqrt{y}} + \sqrt{y} = 10 - \sqrt{x}$

$\sqrt{x^5} + \sqrt{y^5} = 275$

$x = 9$, or 4, or $\dfrac{-13 \pm \sqrt{-51}}{2}$

$y = 4$, or 9, or $\dfrac{-13 \mp \sqrt{-51}}{2}$.

1. *A and B enter into a speculation for which they jointly invest Rs. 500. After 4 months A withdraws from the partnership and receives as his share, capital and profit, Rs. 450; 6 months later B finds that his share, capital and profit, is the same. What was the stock of each?*

Let the stock of A be x, and that of B, y,

then $\quad x + y = 500 \quad \therefore \quad y = 500 - x$;

as A lets his money remain 4 months and B 10, the sum of the relative value of their stocks is $10x + 4y$,

and the total gain, $450 + 450 - 500 = 400$,

$$10x + 4y : 400 :: 10x : \frac{4000x}{10x + 4y} = A\text{'s profit.}$$

A's stock and profit is then expressed by $x + \dfrac{2000x}{5x + 2y} \quad 450$,

or $\quad 5x^2 + 2xy + 2000x = 2250x + 900y.$

substituting the value of $y = 500 - x$ from the 1st equation, we have

$$5x^2 + 2x(500 - x) \quad 250x = 900(500 - x)$$

or $\quad 3x^2 + 1650x = 450000$

or $\quad + 550x = 150000$

$$x = -275 \pm \sqrt{75625 + 150000}$$

$$\therefore \quad x = -275 + 475 = 200$$

$$\therefore \quad y = 500 - 200 = 300.$$

2. *A baboo sold his horse for Rs. 90, by which he lost as much per thousand as the horse cost him. What did he pay for the horse?*

Let x be the price of the horse

$$1000 \quad x : \cdot x : \frac{x^2}{1000} = \text{the loss,}$$

but the price he paid for the horse, must be equal to, what he received added to the loss, which is expressed by

$$90 + \frac{x^2}{1000}$$

or $\quad x^2 - 1000x \quad - 90000$

$$x = 500 \pm \sqrt{250000 - 90000}$$

$$x = 900 \text{ or } 100.$$

He paid for the horse either Rs. 900 or Rs. 100.

3. *The costs of the last annual fireworks at the Hindu College, amounting to Rs. 1230, were to be paid by the subscriptions of teachers and pupils, but before their exhibition took place and the subcriptions were all collected, twelve of the*

pupils, who happened to have subscribed an average share, left college, each subscriber had then to pay one anna more to supply the deficiency. How many did at first subscribe?

Let x be the number,

then $\frac{1230}{x}$, denotes an average subscription

and $\frac{1230}{x-12}$ will be the sum each on an average had to pay after twelve individuals had left college,

we have then the equation

$$\frac{1}{16} = \frac{1230}{x-12} - \frac{1230}{x}$$

or

$$x^2 - 12x = 1230 \times 16 \times x - 1230 \times 16 \times x + 1230 \times 12 \times 16$$

or $\quad x^2 - 12x = 236160.$

$$\therefore x = 6 \pm \sqrt{36 + 236160}$$

$$x = 492$$

the negative quantity -480 cannot answer to the condition of the question.

4. *Receiving an order to draw a walk of an equal width round a rectangular garden* a *feet long by* b *broad, whose area is to be equal to that of the garden. What width must I give to the walk?*

Let x be the width of the walk, the area then of the whole walk is $= 2(a + 2x)x + 2bx$; this area must be equal to that of the garden, which is expressed by ab, we therefore have the equation

$$2x(a + 2x) + 2bx = ab$$

or

$$x^2 + \frac{x}{2}(a + b) = \frac{ab}{4}$$

$$x = \frac{-(a + b) \pm \sqrt{(a + b)^2 + 4ab}}{4}.$$

5. *There are three numbers, the difference of whose difference is* 18, *their sum is* 69, *and their continual product* 8160. *What are the numbers?*

Let x represent the greatest,
y the second,
z the least;

then $\quad (x - y) - (y - z) = 18$, by the first condition;

or $\quad x - 2y + z = 18,$

and $x + y + z = 69$ by the second;

by subtraction $3 y = 51$

$$y = 17$$

$$\therefore x + z = 69 - 17 = 52$$

and by the last condition

$$x y z = 8160$$

$$x z = \frac{8160}{17} = 480$$

$$x^2 + 2 x z + z^2 = 2704;$$

subtracting $4 x z = 1920$

$$x^2 - 2 x z + z^2 = 784$$

and extracting the square root, we get

$x - z = 28$, and we found

$$x + z = 52$$

$$\therefore \quad x = 40,\ y = 17, \text{ and } z = 12.$$

6. *The fore wheels of my carriage make 6 revolutions more than the hind wheels in going over 120 yards, but if the circumference of each wheel be increased one yard, they will make only 4 revolutions more in going over the same distance. Required the circumference of each?*

Let x represent the circumference of one of the hind wheels,
y that of one of the fore wheels,
then by the first condition

$$\frac{120}{y} - \frac{120}{x} = 6$$

or $20 x - 20 y = x y \quad (A)$;

and by the second condition

$$\frac{120}{y + 1} - \frac{120}{x + 1} = 4$$

or $30 (x + 1) - 30 (y + 1) = (x + 1)(y + 1)$,

or $30 x + 30 - 30 y - 30 = x y + x + y + 1$,

or $29 x - 31 y - 1 = x y\ (A) = 20 x - 20 y$,

$$\therefore \quad 9 x = 11 y + 1$$

$$\therefore \quad x = \frac{11 y + 1}{9} \qquad (B)$$

substituting this value of x in equation (A)

$$\frac{220 y + 20}{9} - 20 y = \frac{11 y^2 + y}{9}$$

or $$11\,y^2 + y = 220\,y + 20 - 180\,y$$

$$11\,y^2 - 39\,y = 20$$

$$y^2 - \frac{39}{11}\,y = \frac{20}{11}$$

$$y = \frac{39}{22} \pm \frac{1}{22}\sqrt{39^2 + 44 \times 20}$$

$$y = \frac{39 \pm 49}{22} = 4 \text{ or } - \frac{5}{11}:$$

substituting these values of y in equation (B)

$$x = \frac{44 + 1}{9} = 5, \text{ and}$$

$$\frac{-11 \times \frac{5}{11} + 1}{9} = \frac{-4}{9}$$

7. *A Gentleman visiting a charity school gave to each child one rupee and Rs. 50 more to him who was the first to solve an Algebraic problem; on a second visit he found, that the number of children had increased by three; to each he made on that occasion a present of Rs. 15, and to the best he gave Rs. 5 more. He then found to have distributed equal sums on each visit. How many children were there at school?*

Let x represent the number of children on the first visit; on the first occasion the number of rupees he distributed is indicated by $$x^2 + 50,$$
and that on his second visit by
$$15\,(x + 3) + 5$$
which by the condition expressed in the question must be equal; we have then the equation:

$$x^2 + 50 = 15\,(x + 3) + 5$$

or $$x^2 + 50 = 15\,x + 50$$

$$x^2 - 15\,x = 0$$

$$x = \frac{15}{2} \pm \frac{1}{2}\sqrt{15^2}$$

$$= \frac{15 \pm 15}{2} = 15 \text{ or } 0.$$

This equation may be solved like one of the first degree, for resuming the equation

$$x^2 - 15\,x = 0$$

we may write

$$x^2 = 15\,x$$

dividing by x we get

$$x = 15$$

8. *What number is that, which being divided by the product of its digits diminished by the right hand digit, the quotient is 6, and if 45 be subtracted from it, the digits will be inverted?*

Let x and y be the digits

$$\therefore\ 10\,x + y = \text{the number}$$

and

$$\frac{10\,x + y}{x\,y - y} = 6 \qquad (A)$$

or

$$10\,x + y = 6\,x\,y - 6\,y$$

$$10\,x = 6\,x\,y - 7\,y.$$

By the second condition

$$10\,x + y - 45 = 10\,y + x$$

or

$$9\,x - 9\,y = 45$$

$\therefore$

$$x - y = 5$$

or

$$x = 5 + y; \qquad (B)$$

substituting this value of x in equation (A)

$$\frac{50 + 10\,y + y}{5\,y + y^2 - y} = 6,$$

or

$$\frac{50 + 11\,y}{y^2 + 4\,y} = 6,$$

or

$$6\,y^2 + 24\,y = 50 + 11\,y$$

$$y^2 + \frac{13}{6}\,y = \frac{50}{6}$$

$$y = -\frac{13}{12} \pm \frac{1}{12}\sqrt{13^2 + 50 \times 24}$$

$$\therefore\quad y = -\frac{13 \pm 37}{12} = 2 \text{ or } -\frac{25}{6}$$

and $x = (B)\ 5 + 2 = 7$

or

$$5 - \frac{25}{6} = \frac{5}{6}.$$

9. *Playing with a number of balls, I found to have just sufficient to form an equilateral triangle, three deep; and if 597 be taken away, the remainder will form a hollow square four deep, the front of which contains one ball more than the square root of the number contained in the front of the triangle. What is the number of balls?*

Let x stand for the number of balls in front of the triangle,

then $3(x-1)$ is the perimeter of the outward row,

$3(x-4)$ the perimeter of the 2nd row,

$3(x-7)$ the perimeter of the 3rd row

(*A*) $\therefore\ 9x-36 =$ the number of balls forming the equilateral hollow triangle: the number of balls required to form the hollow square is, by the second condition

$$9x-36-597, \text{ or } 9x-633 \quad (B)$$

and by the 3rd hypothesis the front of the square is

$$\sqrt{x}+1$$

$4\sqrt{x}$ will then express the perimeter of the 1st row,

$4(\sqrt{x}-2)$ that of the 2nd row,

$4(\sqrt{x}-4)$ that of the 3rd row,

$4(\sqrt{x}-6)$ that of the 4th row,

$16\sqrt{x}-48$, must therefore express the number of balls forming the hollow square. Equalising this quantity with equation (*B*) which expresses the same thing, we have

$$9x-633 = 16\sqrt{x}-48$$

or
$$9x-16\sqrt{x} = 585$$

$$x-\frac{16\sqrt{x}}{9} = \frac{585}{9}.$$

$$\sqrt{x} = \frac{8}{9} \pm \frac{1}{9}\sqrt{64+585\times 9}$$

$$\therefore\ \sqrt{x} = \frac{8\pm 73}{9} = 9$$

$$\therefore\ x = 81$$

which value of x substituted in equation (*A*) gives
$9\times 81-36 = 693 =$ the whole number of balls.

10. *On visiting a school, I found the young men of first rate and inferior abilities promiscuously seated on three benches; the first held the third part of the school plus one and a third student; the second, half the school less five and a half students, and the third bench held one fourth of the school plus one and a quarter students. Proposing a geometrical question, the third part of those that were seated on the first bench minus one third of a student; half those minus one on the second bench; and a quarter of those on the third bench plus three and a half students answered the question correctly; to them alone I awarded prizes. I gave to those on the first bench as many Rupees as there were pupils in the schools plus Rs.* 9; *to those on the second bench, twice as many Rupees as there were pupils in the school less Rs.* 15, *and to those on*

the third bench twice the number of Rupees that there were pupils in the school less Rs. 4. The prizes to each pupil being found to be equal, it is required to determine the number of pupils in the school?

Let x be the number of students in the school:

on the 1st bench then, were seated $\frac{x}{3} + 1\frac{1}{3} = \frac{x+4}{3}$

2nd ... ... $\frac{x}{2} - 5\frac{1}{2} = \frac{x-11}{2}$

3rd ... ... $\frac{x}{4} + 1\frac{1}{4} = \frac{x+5}{4}$

Those who solved the problem,
- on the 1st bench, $\frac{1}{3} \times \frac{x+4}{3} - \frac{1}{3} = \frac{x+1}{9}$
- 2nd $\frac{1}{2} \times \frac{x-11}{2} - 1 = \frac{x-15}{4}$
- 3rd $\frac{1}{4} \times \frac{x+5}{4} + 3\frac{1}{2} = \frac{x+61}{16}$

Prizes given to
- the 1st bench, $x + 9$
- the 2nd bench, $2x - 15$
- the 3rd bench, $2x - 4$

but the prize of each pupil may be expressed by the price awarded to a bench divided by the number of scholars that solved the problem on that bench, we have then

Prize to each scholar,
- on the 1st bench, $\dfrac{x+9}{\frac{x+1}{9}} = \dfrac{9x+81}{x+1}$
- 2nd $\dfrac{2x-15}{\frac{x-15}{4}} = \dfrac{8x-}{x-}$
- 3rd $\dfrac{2x-4}{\frac{x+61}{16}} = \dfrac{32x-64}{x+61}$

and as these prizes are equal, we have

$$\frac{9x+81}{x+1} = \frac{8x-60}{x-15} = \frac{32x-64}{x+61}$$

taking the two first of these quantities, we get

$$9x^2 + 81x - 135x - 1215 = 8x^2 - 60x + 8x - 60$$

$$x^2 - 2x = 1155$$

$$x = 1 \pm \sqrt{1 + 1155} = 1 + 34$$

$\therefore x = 35 =$ the number of pupils in the school.

11. *A gentleman bought a number of stuffed China birds, for Rs. 72; had he bought 6 less for the same money he would have paid R. 1 more for each. How many did he buy, and what was the average price of each?*

Let x be the number of birds he bought

$$\frac{72}{x} = \text{the price of one,}$$

$$\frac{72}{x-6} = \text{the price of one if he had bought 6 less.}$$

$$\therefore \quad \frac{72}{x-6} - \frac{72}{x} = 1,$$

or $$72\,x - 72\,x + 432 = x^2 - 6\,x,$$

$$x^2 - 6\,x = 432$$

$$x = 3 \pm \sqrt{9 + 432}$$

$$\therefore \quad x = 3 \pm 21 = 24.$$

12. *My kansamah brought from the bazar* 150 *potatoes and* 80 *small fish for* 14 *annas. He got for* 2 *annas* 10 *potatoes less than he got fish for* 6 *annas. What was the price of the potatoes and that of the fish?*

Let the price of the potatoes be x

... ... fish y

$$\therefore \quad 150\,x = \text{the price of all the potatoes,}$$

$$80\,y = \quad \ldots \quad \ldots \quad \text{fish,}$$

$$\therefore \quad 150\,x + 80\,y = 14$$ by the 1st supposition

$$\therefore \quad x = \frac{14 - 80\,y}{150} \qquad (A)$$

By the 2nd supposition

$$\frac{6}{y} - \frac{2}{x} = 10, \therefore 6\,x - 2\,y = 10\,x\,y$$

substituting the value of x from equation (A)

$$84 - 480\,y - 300\,y = 140\,y - 800\,y^2$$

$$800\,y^2 - 920\,y = -84$$

$$y^2 - \frac{230}{200}\,y = -\frac{21}{200}$$

$$y = \frac{115}{200} \pm \frac{1}{200}\sqrt{(115)^2 - 21 \times 200}.$$

or $$y = \frac{115}{200} \pm \frac{1}{200}\sqrt{9025}$$

$$y = \frac{115 \pm 95}{200} = \frac{210}{200} \text{ or } \frac{20}{200} \quad \frac{1}{10}$$

he got therefore 10 fish for one ana,

and substituting $\frac{1}{10}$ for y in equation (A) we get

$$x = \frac{1}{25}, \text{ or he got 25 potatoes for one ana.}$$

13. *I have two square compounds that I wish to have paved with marble slabs of which each is a cubit square. The side of one compound is* 16 *cubits longer than the other, and both pavements will contain* 3490 *marble slabs. How many cubits was the side of each compound?*

Let x and $x + 16$ represent the number of cubits in the sides of each compound

then $\quad x^2 + (x + 16)^2 =$ the number of marble slabs

$$2x^2 + 32x + 256 = 3490$$

or

$$x^2 + 16x = 1617$$

$$\therefore \quad x = -8 \pm \sqrt{64 + 1617} = -8 \pm \sqrt{1681}$$

$$\therefore \quad x = -8 \pm 41 = 33$$

and the side of the greater compound was $33 + 16 = 49$ cubits.

14. *My master gave me turkeys, geese and ducks to take to the bazar, where I sold each bird for as many Rupees as there were birds of that kind. The money I got for the turkeys multiplied by that I got for the geese amounted to Rs.* 64; *but multiplying the same by the money I got for the ducks, I had Rs.* 144; *again multiplying the money I got for the geese by the money I got for the ducks I had only Rs.* 36. *How many birds of each kind were there?*

Let x stand for the number of turkeys

y ... ... geese

z ... ... ducks.

As each turkey sold for x Rupees, the money I cleared for all the turkeys was x^2; for a similar reason y^2 and z^2 represent the produce of the sales of the geese and ducks.

Then by the second condition

$$x^2 y^2 = 64 \qquad x^2 = \frac{64}{y^2}$$

$$x^2 z^2 = 144 \qquad x^2 = \frac{144}{z^2}$$

$$y^2 z^2 = 36 \qquad z = \frac{6}{y}$$

from $\dfrac{64}{y^2} = \dfrac{144}{z^2}$ we

$$z = \frac{3\,y}{2}.$$

Eliminating z $\dfrac{3\,y}{2} = \dfrac{6}{y}$

$$3\,y^2 = 12$$

$$y = 2$$

$$\therefore\ x = 4 \text{ and } z$$

The general enunciation of this problem would be :

15. *There are three square numbers of which the product of the 1st by the 2nd, the product of the 2nd by the 3rd, and the product of the 1st by the 3rd give respectively the numbers* a, b, c. *What are they?*

$$x^2\,y^2 = a$$

$$y^2\,z^2 = b$$

$$x^2\,z^2 = c$$

$$\therefore \qquad x = \sqrt[4]{\frac{a\,c}{b}}\,;\ y = \sqrt[4]{\frac{a\,b}{c}}\,;\ z = \sqrt[4]{\frac{b\,c}{a}}.$$

16. *I bought two pieces of Gros de Naples, one measured $6\frac{1}{2}$ ells more than the other, and some cases of claret; I had to pay for each ell of Gros de Naples as many rupees as I bought cases of claret, and for the claret I paid as many anas per case as there were ells of Gros de Naples in both pieces. Disposing of my purchase I gained 12 per cent. and received for the whole Rs. 476. On the following day I bought as many cases and one more of the same claret as I had bought ells of Gros de Naples, and sold them again for Rs. 486, by which I gained 20 per cent. How many ells did each piece of Gros de Naples contain, and how many cases of claret did I buy?*

Let x represent the number of ells in both pieces
y cases of claret.

Then $x\,y$ will be the price of both pieces of Gros de Naples

and $\dfrac{xy}{16}$ all the cases of claret.

The value of both was $x\,y + \dfrac{x\,y}{16} = \dfrac{17\,x\,y}{16}$

by gaining 12 per cent. on it I got Rs. 476, we have then the equation,

$$\frac{17}{16}xy + \frac{12}{100} \times \frac{17}{16}xy = 476$$

$$\frac{17}{16}xy + \frac{204}{1600}xy = 476$$

$$1904\,xy = 761600$$

or $$xy = 400.$$

My next purchase was $x + 1$ cases of claret, for each I paid as [illegible]re $\frac{x}{16}$ Rs.; I paid then for my 2nd purchase $(x+1)\frac{x}{16}$ for which with the profit of 20 per cent. I received Rs. 486, which gives the equation

$$(x+1)\frac{x}{16} + (x+1)\frac{x}{16} \cdot \frac{20}{100} = 486$$

or $$(x+1)\,x + (x+1)\,x.\frac{1}{5} = 7776$$

or $$6x^2 + 6x = 38880$$

$$x^2 + x = 6480$$

$$\therefore x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 6480 \times 4}$$

$$\therefore x = \frac{-1 \pm 161}{2} = 80$$

the lesser piece therefore contained $36\frac{3}{4}$ ells, and the greater $43\frac{1}{4}$; and from $xy = 400$ or $y = \frac{400}{x} = \frac{400}{80} = 5 =$ the number of cases of claret he bought on the first day, and $x + 1$ the number he bought on the second day, or 81 cases.

17. *It is required to find two numbers, such that their sum, product, and difference of their squares, shall be equal.*

Be $x =$ the greater number, and $y =$ the less.

$$x + y = xy = x^2 - y^2.$$

Divide the last quantity by its equal the first, we get

$$\frac{x^2 - y^2}{x + y} = 1\,;$$

but without taking into consideration the equality of these two expressions, we have generally

$$\frac{x^2 - y^2}{x + y} = x - y$$

$$\therefore x - y = 1 \text{ or } x = y + 1.$$

Substituting this value of x in the 1st and 2nd expressions

$$y + 1 + y = (y + 1)\,y$$

$$\text{or } y^2 - y = 1$$

$$y = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4}$$

$$\therefore y = \frac{1 \pm \sqrt{5}}{2}$$

and
$$x = y + 1 = \frac{1 \pm \sqrt{5}}{2} + 1$$

$$\therefore x = \frac{3 \pm \sqrt{5}}{2}$$

18. *In a foundry they cast a certain number of cannon of two different calibre. The first sort weighed as many cwt. as they cast cannon of that sort; of the second sort they cast as many as two pieces of both sorts weighed cwt. and yet there were used 31 cwt. of metal more for this than for the 1st sort. Subsequently they received order to cast $\frac{1}{5}$th the number of cannon of the first, and two more of the second than of the first sort, but to add 5 cwt. of metal more to each of the two sorts; they now used 3 cwt. of metal more for the first sort. How many cannon of each calibre were cast?*

x = number of cwt. of each of 1st kind

y = ditto, 2nd ditto,

then x^2 indicates the quantity of metal used for all the cannon of the 1st sort; the number of the 2nd sort is $x + y$; the metal used for all the cannon of this sort, is expressed then by

$$y\,(x + y)$$

but as this quantity of metal weighs 31 cwt. more than the metal used for the cannon of the 1st sort, we may express this condition by the following equation:

$$x^2 = y\,(x + y) - 31$$

or
$$x^2 = y^2 + x\,y - 31 \qquad (A.)$$

By a similar reasoning the condition of second order for cannon is expressed by

$$(x + 5)\,\frac{x}{5} = (y + 5)\left(\frac{x}{5} + 2\right) + 3$$

or
$$\frac{x^2}{5} + x = \frac{x\,y}{5} + 2\,y + x + 10 + 3$$

or
$$x^2 = x\,y + 10\,y + 65 \qquad (B.)$$

Eliminating x^2, by equalising the equations A and B, we have

$$y^2 + x\,y - 31 = x\,y + 10\,y + 65$$
$$y^2 - 10\,y = 96$$
$$\therefore y = 5 \pm \sqrt{25 + 96}$$
$$\therefore y = 5 \pm 11 = 16$$

Substituting this value of y in equation (B) we get

$$x^2 - 16\,x = 160 + 65$$

or

$$x^2 - 16\,x = 225$$
$$\therefore x = 8 \pm \sqrt{64 + 225}$$
$$\therefore x = 8 \pm 17 = 25$$

19. *I bought 2 casks of brandy for Rs. 114, one of which contained 6 gallons more than the other, and the price per gallon was 5 two anas pieces less than one-seventh of the number of gallons in the less. Required the price per gallon and the number of gallons in each cask.*

If x represents the number of gallons in the less cask,

$x + 6$ represents that in the greater,

and $2\,x + 6 =$ the total number of gallons;

the price of each gallon will then be

$$\frac{x}{7} - \frac{5}{8}$$

consequently, the cost of the whole number of gallons

$$(2\,x + 6)\left(\frac{x}{7} - \frac{5}{8}\right) = 114$$

or

$$\frac{2\,x^2}{7} - \frac{10\,x}{8} + \frac{6\,x}{27} - \frac{30}{8} = 114$$

or

$$16\,x^2 - 70\,x + 48\,x - 210 = 6384$$
$$16\,x^2 - 22\,x = 6594$$
$$\therefore x^2 - \frac{22}{16}\,x = \frac{6594}{16}$$
$$x = \frac{11}{16} \pm \frac{1}{16}\sqrt{121 + 6594 \times 16}$$
$$x = \frac{11 \pm 325}{16} = 21 \text{ gallons}$$

the price of each $\frac{x}{7} - \frac{5}{8} = \frac{21}{7} - \frac{5}{8} = 3 - \frac{5}{8} = 3 - \frac{10}{16}$

$=$ **Rs. 3 — anas 10, = Rs. 2 6 anas.**

20. *I have two vessels of which the first is 2 inches longer, 2 inches narrower and half an inch deeper than the second. When both are filled, the first contains 81 cubic inches more than the other, but when only partly filled wanting one inch, then the first contains 83 cubic inches more. The total surface of the 1st vessel has 28 square inches more than the total surface of the second. The volume of a third vessel, whose bottom is a square, of which the side has 2 inches more than twice the depth of the second vessel, with a depth equal to the breadth of the same vessel, is 25 cubic inches less than that of the two first vessels together. Required the dimensions of the 3 vessels.*

Expressing the length, breadth and depth of the 2nd vessel by $x, y, z,$
that of the 1st vessel by $x + 2, y - 2, z + \frac{1}{2}$
Volume of 1st vessel $= (x + 2)(y - 2)(z + \frac{1}{2}) - 81 = x\,y\,z$
the volume of the second.
Again, $(x + 2)(y - 2)(z - \frac{1}{2}) - 83 = x\,y\,(z - 1)$.

By reduction we get

$$x = y - 1.$$

The surface of 1st vessel
$2\,(x + 2)(y - 2) + 2\,(x + 2)\,(z + \frac{1}{2}) + 2\,(y - 2)\,(z + \frac{1}{2}) - 28$
$=$ (the surface of 2nd) $2\,x\,y + 2\,x\,z + 2\,y\,z$

By substituting the value of $x = y - 1$ into this equation, z vanishes, and we get for

$$y = 14.$$

The last condition is expressed, by
$(2\,z + 2)^2 y + 25 = x\,y\,z + (x + 2)(y - 2)(z + \frac{1}{2})$
substituting the values found for x and y, and reducing, we get

$$56\,z^2 - 250\,z = 9$$

$$\text{or } z = \frac{125}{56} \pm \frac{1}{56}\sqrt{125^2 + 9 \times 56}$$

$$\therefore z = \frac{125 \pm 127}{56} = 4\tfrac{1}{2}.$$

21. *Being asked respecting my monthly salary, I answered "If I add Rs. 40 to it, and also subtract Rs. 60 from it, and take the squares of the numbers thus obtained, the difference of these squares is more than the square of half my salary added to twelve times my salary, by Rs. 24,880." How much do I receive per month?*

Ans. Rs. 560.

22. *Two merchants* A *and* B *jointly invested Rs.* 2000 *in business.* A, *lets his money remain* 17 *months, and received back in capital and profit Rs.* 1710; B, *allowed his money to remain* 12 *months, and received in capital and interest Rs.* 1040. *How much did each advance?*

Ans. A, *Rs.* 1200, B, *Rs.* 800.

23. *A horse-dealer buys a horse, and pays a certain sum for it, he afterwards sells it again for Rs.* 171, *and gains exactly as much per cent. as the horse had cost him. How much did he pay for the horse?*

Ans. Rs. 90.

24. *A cask, whose contents are* 20 *gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture. and it appears, that if* $6\frac{2}{3}$ *gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.*

Ans. 10 *gallons.*

25. *A ship containing* 74 *sailors, and a certain number of soldiers besides officers, took a prize. The sailors received each one-third as many rupees as there were soldiers, and the soldiers received Rupees* 3 *a piece less, and Rupees* 768 *fell to the share of the officers. Had the officers received however, nothing, the soildiers and sailors might have received half as many rupees per man, as there were soldiers. How many soldiers were there, and how much did each receive?*

Ans. There were 36 *soldiers, each soldier received Rs.* 9, *and each sailor Rs.* 12.

26. *Two men,* A *and* B, *undertake to perform a piece of work in four days, for which they are to receive a certain number of rupees; but, after some time, finding that they shall not be able to finish it in the time proposed, they call in* C *to assist them, and upon an equitable division of the money,* C *receives a sum equal to the square root of the whole number of rupees, but had they been obliged to call in* C *to their assistance* $1\frac{1}{9}$ *day sooner, his share of the money would have been two-fifths more. How long did* C *work, and what did he receive?*

Ans. He worked 2 *days, and received Rs.* 5.

27. *Two women carried to the bazar a certain number of mangoes, one however had* 30 *mangoes more than the other, for which they jointly receive Rs.* 13. *"I should have received," said the first to the other, "at my price, Rs.* 7 : 8 *for your mangoes." "I must admit," the other answered*

her, "that at my low price I should have got no more than Rs. 5 : 10 *for your mangoes." How many mangoes had each?*

Ans. The one 150, *the other* 180 ; *or the one* 294, *the other* 324.

28. A, B *and* C *wished to buy a house, but neither had money enough for the purpose.* A *begged of* B *and* C *the third part of their money in order to enable him to buy it ; on the other hand,* B *asked* A *and* C *for the fourth part only of their money to enable him to buy it for himself. On which* C *said, if* A *lends me only the 5th part of his money and* B, *the third part of his minus Rs.* 20.—*I can buy it alone.* A *and* B *agreeing,* A *said, that he should now content himself with buying the garden attached to it, valued for as much as the square root of what all three possessed added to Rs.* 53,97—*which he could pay with the sixth of his money. How much money had each?*

Ans. A, *Rs.* 600, B, *Rs.* 768 : 12 *and* C, *Rs.* 750.

29. *A trader had on hand three chests, each containing an equal quantity of Hyson tea of the same sort. On one, being slightly damaged, he lost Rs.* 100, *on another, he gained Rs.* 100 ; *and on the third, by selling it per lb., for as many rupees as the 20th part of the number of lbs. it contained, he gained Rs.* 120. *Had he been able to sell the whole quantity at the price he sold the last chest, he would have gained* 20 *per cent.*

How many lbs. did a box contain? at how much per lb. did he buy the tea? and at what price per lb. did he dispose of each chest?

Ans. A chest contained lbs. 120.

he paid for the tea per lb.	*Rs.*	5	0	0
of the 1st chest, he sold the lb. for.........		4	2	8
2nd		5	13	4
3rd		6	0	0

30. *A shopkeeper sold* 320 *seers of pepper, and* 175 *seers of sugar for Rs.* 135 ; *but he sold* 4 *seers more of sugar for Rs.* 4, *than he did pepper for Rs.* 5. *What was the price of a seer of each?*

Ans. Pepper 5 *anas a seer.*
Sugar 5 *seers for one rupee.*

31. A *and* B *were going to market, the first with cucumbers and the second with three times as many eggs ; and they find that if* B *gave all his eggs for the cucumbers,* A *would lose* 10 *pice, according to the rate at which they were then selling.* A *therefore reserves* $\frac{2}{5}$*ths of his cucumbers ; by which* B *would lose six pice, according to the same rate. But* B, *selling the cucumbers at* 6 *pice a piece, gains upon the whole the price of six eggs. Required the number of eggs and cucumbers, and their price.*

Ans. 30 *eggs,* 10 *cucumbers. Price of an egg* 1 *pice, and that of a cucumber* 4 *pice.*

32. *What 5 numbers possess these properties, that if each, beginning with the 1st, be multiplied by the one which succeeds (follows) it, but the last again by the 1st, the products* a, b, c, d, e, *are obtained?*

$$Ans.\ \sqrt{\frac{ace}{bd}},\ \sqrt{\frac{abd}{ce}},\ \sqrt{\frac{bce}{ad}},\ \sqrt{\frac{acd}{be}},\ \text{and}\ \sqrt{\frac{bde}{ac}}.$$

33. *But if, instead of* 5, *seven numbers be required, and the products be* a, b, c, d, e, f, g; *what numbers are they then?*

$$Ans.\ \sqrt{\frac{aceg}{bdf}},\ \sqrt{\frac{abdf}{ceg}},\ \text{\&c. \&c.}$$

34. *There are two numbers, one of which is greater than the other by* 8, *and whose product is* 240. *What numbers are they?*

Ans. 12 *and* 20.

35. *The sum of* 2 *numbers* = a, *their product* = b. *What numbers are they?*

$$Ans.\ \frac{a + \sqrt{(a^2 - 4b)}}{2}\ \text{and}\ \frac{a - \sqrt{(a^2 - 4b)}}{2}.$$

36. *It is required to find a number, whose square exceeds its simple power by* 306?

Ans. 18.

37. *It is required to find a number, such, that if we multiply its 3rd part by its 4th, and to the product add* 5 *times the number required, this sum exceeds the number* 200 *by as many as the number sought is less than* 280?

Ans. 48.

38. *A person who was asked his age, answered, "My mother was* 20 *when I was born; her age multiplied by mine, exceeds our united ages by* 2500." *What was his age?*

Ans. 42.

39. *Determine the fortunes of* 3 *persons,* A, B, C, *from the following data:—For every* 5 *rupees which* A *possesses,* B *has* 9, *and* C 10. *Farther, if we multiply* A's *money (expressed in rupees, and considered merely as a number) by* B's, *and* B's *money by* C's, *and add both products to the united fortunes of all* 3, *we shall get* 8832. *How much had each?*

Ans. A, 40, B, 72, C, 80.

40. *A person buys some pieces of cloth at equal prices for* 60 *rupees. Had he got* 3 *more pieces for the same sum, each piece would have cost him* 1 *rupee less. How many pieces did he buy?*

Ans. 12.

41. *A charitable person divides a sum of 90 Rs. in equal shares amongst the poor of a small town. But as 30 of those whom he thought of relieving stood no longer in need of assistance, each of the remaining paupers had for his share 4 annas more than he otherwise would have had. How many paupers were there at first?*

Ans. 120.

42. *A person dies, leaving children and a fortune of* 46,800 *rupees which, by the will, is to be divided equally amongst them. It happens, however, that immediately after the death of the father, two of his children also die. If, consequently, each child receives* 1950 *Rs. more than he or she was entitled to by the will, how many children were there?*

Ans. 8 *children.*

43. *Required to find a number, such, that if a given number* c *be divided by it, and also by a number greater than it by* a, *the difference of the two quotients* = d. *What number is it?*

$$Ans.\ -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + \frac{ac}{d}\right)}$$

44. *Twenty persons, men and women, together spend* 48 *Rs. at a tavern; viz. the men* 24 *Rs., and the women the same sum. Now, on inspecting the bill it is found, that the men have to pay* 1 *R. each more than the women. How many men, therefore, were there in company?*

Ans. 8.

45. *Two travellers,* A *and* B, *set out at the same time from two different places,* C *and* D, A *from* C *to* D, *and* B *from* D *to* C. *It appears that* A *has already gone* 30 *miles more than* B, *and according to the rate at which they travel,* A *calculates that he can reach the place* D, *in* 4 *days, and that* B *can arrive at the place* C *in* 9 *days. What is the distance between* C *and* D?

Ans. 150 *miles.*

46. *In the preceding problem, let* d *be the distance which* A *had travelled more than* B; a *the time which* A *requires to finish the remainder of his journey, and* b *the time which* B *requires, in order to finish his. What expression will give the distance between* C *and* D?

$$Ans.\ \frac{d(\sqrt{b} + \sqrt{a})}{\sqrt{b} - \sqrt{a}}.$$

47. *There are two numbers,* a *and* b, *given; it is required to divide each of them into two such parts, that the one part of*

a *is to one part of* b, *as* m *to* n, *and that the product of the other two parts* = p. *How must they be divided?*

Ans. Let $\frac{na + mb\sqrt{[(na - mb)^2 + 4mnp]}}{2mn} = A$,

then 1 *part of* a = m A, *and one part of* b = n A.

48. *Again. Let it be required, as in the preceding problem, to divide the two numbers* a, b, *so that the first parts may be to one another as* m *to* n, *but the sum of the squares of the two others* = s. *How then must they be divided?*

Ans. Let $\frac{am + bn \pm \sqrt{\{(m^2 + n^2)s - (an - bm)^2\}}}{m^2 + n^2} =$

A, *then the* 1*st part of* a = m A, *and the* 1*st part of* b = n A.

49. *It is required to find a number, consisting of* 3 *digits, such, that the sum of the squares of the digits, without considering their position, may be* = 104; *but the square of the middle digit exceeds twice the product of the other* 2 *digits by* 4; *farther, that if* 594 *be subtracted from the number sought, the* 3 *digits become inverted?*

Ans. 862.

We must not always immediately infer that the quantities required to be found, are the unknown quotients in the calculation; we should otherwise not unfrequently hit upon higher equations than are necessary for the solution of the problem, and this we must try to avoid as much as possible. It is often better to seek first any combination of the quotients, as, for instance, the sum, the difference, the product, the sum of the squares, the difference of the squares, and so on, and hence to determine the quotients themselves. As this is a very important point in Algebra, and one which cannot be too well observed, we shall now give a tolerable number of such problems; more of this kind will occur in the examples of geometrical progression.

50. *Find two numbers, whose difference, multiplied by the difference of their squares,* = 160; *and whose sum, multiplied by the sum of their squares, gives the number* 580.

Ans. The sum of the two numbers is 10, *and their product* 21; *therefore the numbers themselves are* 3 *and* 7.

51. *A person wishes to find two numbers such, that the sum and product of the numbers together amount to* 34, *and the sum of their squares exceeds the sum of the numbers themselves by* 42. *What numbers are they?*

Ans. The sum of both numbers is 10, *their product* 24, *and the numbers themselves are consequently* 4 *and* 6.

52. *If, in order to make the foregoing problem more gene-*

ral, a *be put instead of* 34, *and* b *instead of* 42: *in this case, by what form will the numbers sought be expressed?*

Ans. *Let* $-1 \pm \sqrt{(4b + 8a + 1)} = 2A$; $2A + 1 \mp \sqrt{(4b + 8a + 1)} = 2B$; *then the two numbers sought are* $\frac{A + \sqrt{(A^2 - 4B)}}{2}$, $\frac{A - \sqrt{(A^2 - 4B)}}{2}$

53. *What two numbers are they, whose sum* a, *and the sum of whose fourth powers* = b?

Ans. *Call the difference of the two numbers sought* d, *then* d $= \sqrt{\{-3a^2 \pm \sqrt{(8a^4 + 8b)}\}}$; *the numbers themselves are consequently* $\frac{a+d}{2}$, $\frac{a-d}{2}$.

54. *The sum of two numbers is* = a, *the sum of their 5th powers* = b. *What numbers are they?*

Ans. The product p *of both numbers is* $= \frac{1}{2}\left\{a^2 \pm \sqrt{\frac{a^5 + 4b}{5a}}\right\}$; *therefore the numbers themselves are*
$\frac{1}{2}\{a + \sqrt{(a^2 - 4p)}\}$, $\frac{1}{2}\{a - \sqrt{(a^2 - 4p)}\}$, or

$$\frac{1}{2}\left\{a \pm \sqrt{-a^2 \pm 2\sqrt{\frac{a^5 + 4b}{5a}}}\right\}$$

55. *The sum of two numbers is* = a, *their products multiplied by the sum of their squares is* = b. *What numbers are they?*

Ans. *Let the products of the two numbers* $= p$, *if* $p = \frac{1}{4}[a^2 \pm a^4 - 8b]$, *then the numbers themselves are* $\frac{1}{2}\{a + \sqrt{(a^2 - 4p)}\}$; $\frac{1}{2}\{a - \sqrt{(a^2 - 4p)}\}$.

56. *The sum of two numbers added to the sum of their squares* = a, m *times the sum of their squares added to* n *times the product of the numbers* = b. *What numbers are they?*

Ans. *The sum* s *and the product* p *of both numbers are expressed by the equations* $n s^2 + (n - 2m) s = 2b + (n - 2m) a$; $2p = s^2 + s - a$. *Having determined from these* s *and* p, *then both the numbers themselves may be found, by solving the equation* $x^2 - sx + p = 0$. *Each number, therefore, has four values.*

57. *In a geometrical proportion the sum of the means* = a, *the sum of the two extremes* = b, *and the sum of the squares of all four terms* = c. *What is the proportion?*

Ans. *The product of the two means, consequently of the two extremes* $= \frac{a^2 + b^2 - c}{4}$, *therefore the required proportion is*
$\frac{1}{2}[b - \sqrt{(c - a^2)}] : \frac{1}{2}[a - \sqrt{(c - b^2)}] = \frac{1}{2}[a + \sqrt{(c - a^2)}] : \frac{1}{2}[b + \sqrt{(b - a^2)}]$.

58. *The difference between the means of a geometrical proportion* = a, *the difference of the two extremes* = c, *and the sum of the squares of all the four terms* = c. *What is the proportion?*

Ans. *The product of the two extremes, or of the two means,* $c - a^2 - b^2$ $\therefore$ *the proportion itself is*

$: \frac{1}{2}[-b + \sqrt{(c - a^2)}] : \frac{1}{2}[-a + \sqrt{(c - b^2)}] = \frac{1}{2}[+a + (c - b^2)] : \frac{1}{2}[b + \sqrt{(c - a^2)}]$.

59. *In a geometrical proportion the product of the two extremes, or means,* = a, *the sum of all four terms* = b, *and the sum of their squares* = c. *Find the proportion.*

Ans. *For shortness-sake, let* $\pm \sqrt{(8a + 2c - b^2)} = A$; *then* $\frac{b - A}{2}$ *is the sum of the two means, and* $\frac{b + A}{2}$ *the sum of the two extremes; consequently the proportion sought is* $\frac{1}{4}[b + A - \sqrt{(2c - 8a + 2bA)}] : \frac{1}{4}[b - A - \sqrt{(2c - 8a - 2bA)}] = \frac{1}{4}[(b - A + \sqrt{(2c - 8a + 2bA)}] : \frac{1}{4}[b + A + \sqrt{(2c - 8a + 2bA)}]$.

60. *The product of the two extremes, or means, of a geometrical proportion* = a, *the difference between the sum of the extremes and the sum of the means* = b, *and the sum of the squares of all four terms* = c. *What is the proportion?*

Ans. *Again; let* $\pm \sqrt{(8a + 2c - b^2)} = A$: *then* $\frac{A - b}{2}$ *is the sum of the means,* $\frac{A + b}{2}$ *the sum of the extremes, and* $\therefore$ *the proportion sought is* $\frac{1}{4}[(A + b - \sqrt{(2c - 8a + 2bA)}] : \frac{1}{4}[A - b - \sqrt{(2c - 8a - 2bA)}] :: \frac{1}{4}[(A - b + \sqrt{(2c - 8a - 2bA)}] : \frac{1}{4}[A + b + \sqrt{(2c - 8a + 2bA)}]$.

When $a = 18$, $b = 2$, $c = 130$, *then* $2 : 3\ 6 : 9$.

When $a = 270$, $b = 20$, $c = 3922$, *then* $5 : 9 :: 30 : 54$.

61. *It is found, by experiment that bodies in falling to the earth, pass through about* 16 *feet in the first second of their motion, and it is known that the spaces passed through, from the commencement of motion, are as the squares of the intervals elapsed. Suppose then that a body be observed to fall through* $\frac{1}{2}$ *of its height during the last two seconds of its descent; required the height from which it fell?*

Ans. 1900, 6 *feet.*

62. *A body was observed to have descended through* 420 *feet in the last one and a half second of time, from what height did it commence its motion? Supposing, as in the preceding example, a body to fall through* 16 *feet in the first second.*

Ans. 1440 *feet.*

Of the solution of quadratic equations, by the tables of tangents and sines.

CXXV. When the numeral parts of a quadratic equation are either large numbers, or complicated fractions, such as examples 73, 74 and 75, page 195, its solution may be more readily obtained by the application of trigonometry, than by the common method, which in this case is very laborious.

Every equation of the second degree is represented by the general formula

$$\text{(A)} \qquad x^2 + p\,x + q = 0$$

whence we get

$$\text{(B)} \qquad x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4\,q}$$

If p and q are large or fractional quantities, the solution of such an equation becomes exceedingly laborious; but trigonometry furnishes prompt and easy means to draw the value of x, either exact or approximate, as the case may be; for should the quantity under the radical not be a perfect square, the value of x cannot possibly be obtained otherwise than by approximation.

CXXVI. Examining the case where the radical is positive and q under the radical, also positive, the equation (B) may then be expressed

$$x = -\frac{p}{2}\left(1 - \sqrt{1 + \frac{4\,q}{p^2}}\right).$$

(C) Supposing tan. $A = \dfrac{2\sqrt{q}}{p}$; (this supposition can not be contested, as a tangent can have every possible value,)

consequently $\dfrac{p}{2} = \dfrac{\sqrt{q}}{tan.\ A} = \dfrac{cos.\ A\sqrt{q}}{sin.\ A}$

$$\therefore \quad x = -\frac{cos.\ A\sqrt{q}}{sin.\ A}\left(1 - \frac{1}{cos.\ A}\right) = \frac{cos.\ A\sqrt{q}}{sin.\ A} \times$$

$$\frac{1 - cos.\ A}{cos.\ A} = \frac{1 - cos.\ A}{sin.\ A}\sqrt{q}$$

$$\text{(D)} \quad \therefore\ x = tan.\ \frac{A}{2}\sqrt{q}$$

By means of the equation (C) we get the value of A, by equation (D) we obtain that of x.

Supposing now that the radical in equation (B) be negative, q under the radical remaining positive, the equation (C) will then become

$$x = -\frac{p}{2}\left(1 + \frac{1}{\cos. A}\right) = -\frac{\cos. A \sqrt{q}}{\sin. A} \times \frac{1 + \cos. A}{\cos. A} =$$

$$-\frac{1 + \cos. A}{\sin. A} \sqrt{q} = -\cot. \frac{A}{2} \sqrt{q}.$$

The second value of x will then be

$$\text{(E)} \qquad x = -\cot. \frac{A}{2} \sqrt{q}.$$

It will be found by the same method, that the preceding formula can also be applied to the case where the particular equation to be solved is of form $x^2 - p\,x - q = 0$. The only difference is that the negative value of x is given by the equation (D), and the positive by equation (E).

CXXVII. Let now the equation (A) to be resolved, be of the following particular form

$$x^2 + p\,x + q = 0;$$

$$\therefore \quad x = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4\,q}.$$

From what has been said, page 164, line 10th, $4\,q$, represented here as negative, must be less than p^2, otherwise the quantity represented here under the radical sign becomes imaginary, and there would exist no real value of x to satisfy the equation.

This premised, considering the radical at first as positive,

$$\text{we have } x = -\frac{p}{2}\left(1 - \sqrt{1 - \frac{4\,q}{p^2}}\right);$$

$$\text{making} \qquad \text{(F)} \quad \sin. A = \frac{2\sqrt{q}}{p}, \quad \text{we get } \frac{p}{2} = \frac{\sqrt{q}}{\sin. A}$$

$$\therefore \quad x = -\frac{\sqrt{q}}{\sin. A}(1 - \cos. A) = -\sqrt{q} \times \tan. \frac{A}{2},$$

here we have the equation (D) with the negative sign in the second member.

Considering now the radical as negative, which does not alter the equation (F); we have $x = -\frac{p}{2}(1 + \cos. A) = -\frac{\sqrt{q}}{\sin. A}$ $(1 + \cos. A) = -\sqrt{q}\cot. \frac{A}{2}$; which is again equation E.

If the equation (A) to be resolved were of the particular form $x^2 - p x + q = 0$ (which would not alter equation (F),) we should find, by the same method as in the preceding cases of negative radicals, that the value of x, is that found by equation (D); and for the case of the positive radical, that the value of x is that given by the equation (E), with the positive sign in the second member.

CXXVIII. The equations (D) and (E) then, give the two values of x in every case; should q (in the second member), be positive? we obtain the arc A by the formula (C), and the two values of x will always be of different signs: but should q, (in the second member,) be negative? the arc A will be found by the formula (F), and both the values of x will be negative, if p is positive in the first member, but both positive if p be negative.

Let it be required to solve the equation

$$x^2 + \frac{9}{35}x = \frac{606564}{853615}.$$

By (C) and (D) we get

$$\tan. A = \frac{70}{9} \cdot \sqrt{\frac{606564}{853615}} \qquad x = \sqrt{\frac{606564}{853615}} . \tan. \frac{A}{2}$$

from	* log.	606564	=	5.7828766	
subtract	log.	853615	=	5.9312620	
	difference of logs.		=	9.8516146	dividing this
difference by 2			=	9.9258073	
	log. 70 . .		=	1.8450980	
	comp. log. 9 . .		=	9.0457575	
log. tan. A			=	0.8166628	= 81°, 19′, 40″
log. tan. $\frac{A}{2}$ or tan. 40° 39′ 50″			=	9.9340133	
the above difference of logs. divided by 2			=	9.9258073	
	log. x		=	9.8598206	= 0,72414.

To ascertain if this approximate value of x, can be expressed exactly by a vulgar fraction, of which both terms may be whole numbers, take the complement of the log. of x, viz. 0,1401794

* NOTE.—For the calculation by logarithms, we must refer the student to a subsequent article (312) headed, *Use of the Logarithmic tables.*

$=$ log. of $\frac{1}{1,38096}$, and if we multiply this fraction by 21, we get $x = \frac{21}{29}$, which is the exact value of x.

Let us now take example 74, page 196:

$$x^2 - \frac{21}{37}x = -\frac{3960}{49247} \quad \text{(see art. 127)}$$

$$\text{by equation (F), sin. } A = \frac{2\sqrt{q}}{p} = 2 \times \frac{37}{21} \cdot \sqrt{\frac{3960}{49247}}$$

$$= \frac{74}{21}\sqrt{\frac{3960}{49247}}$$

$$\therefore x = \text{tan. } \frac{A}{2} \cdot \sqrt{\frac{3960}{49247}}$$

log.	3960		=	3,5976952
„	49247		=	4,6923798
log. q or difference of logs.			=	8,9053154
log.	$\sqrt{q}$	. . . . .	=	9,4526577
log.	74	. . . . .	=	1,8692317
compl. log.	21	. . . . .	=	8,6777807
log. sin. A		. . . .	=	9,9996701 = 87° 46′
log. tan. $\frac{A}{2}$		. . . . . .	=	9,9830673
log.	$\sqrt{q}$	. . . .	=	9,4526577
log.	x	. . . .	=	9,4357250 taking the com-
plement		. . . .	=	$0,5642750 = \frac{1}{3,666} = \frac{3}{11}$

which is the exact value of x.

Taking the next example (75)

$$x^2 - \frac{4}{19}x = \frac{39559}{93347} \quad \text{(see art. 126)}$$

$$\text{tan. } A = \frac{2\sqrt{q}}{p} \therefore \frac{p}{2} = \sqrt{q} \text{ cot. } A$$

$x =$ [illegible] $\left(\frac{1 \pm \cos. A}{\sin. A}\right) \therefore x = \sqrt{q}$ cot. $\frac{A}{2}$ or (see art. 127)

[illegible] tan. $\frac{A}{2}$; in employing first the positive and then the [illegible] sign.

here, $tan.\ A = 2\sqrt{\frac{39559}{93347}} \times \frac{19}{4} = 9.5\sqrt{\frac{39559}{93347}}$

$$\begin{array}{lrl}
\text{log.} & 39559 & -\ 4{,}5972453 \\
\text{„} & 93347 & -\ 4{,}9701004 \\
 & q = & \overline{9{,}6271449} \\
 & \sqrt{q} & -\ 9.8135725 \\
 & 9.5 & -\ 0{,}9777236 \\
 & \text{log. } tan.\ A = & \overline{0.7912961} = 80^\circ\ 48'\ 53''
\end{array}$$

$$cot.\ \frac{A}{2} \text{ or } cot.\ 40^\circ\ 24'\ 26'' = 0{,}0699404$$

$$\sqrt{q}\quad 9{,}8135725$$

$$\sqrt{q}\ cot.\ \frac{A}{2} = 9.8835129 \text{ of which the}$$

$$\text{complement}\qquad 0{,}1164871 = \frac{1}{1{,}6999},$$

multiplying both terms of this fraction by 13, we get

$$x = \frac{13}{17}, \text{ the exact value.}$$

Of the Extraction of the Square Root of Algebraic Quantities.

CXXIX. We have sufficiently illustrated, by the preceding articles, the manner of conducting the solution of literal questions. We have given also an instance of a transformation, namely, that of $\sqrt{a^2 m}$ into $a\sqrt{m}$, page 183, which is worthy of particular attention; since, by means of it, we have been able to reduce the factors, contained under a radical sign, to the smallest number possible, and thus to simplify very much the extraction of the remaining part of the root.

This transformation consists in *taking the roots of all the factors which are squares, and writing them without the radical sign, as multipliers of the radical quantity, and retaining under the radical sign, all those factors, which are not squares.*

This rule supposes, that the student is already able to determine, whether an algebraic quantity is a square, and is acquainted with the method of extracting the root of such a quantity. In order to this, it is necessary to distinguish simple quantities from polynomials.

CXXX. It is evident. from the rule given for the exponents in multiplication, that *the second power of any quantity has an exponent double that of this quantity.*

We have, for example,

$$a^1 \times a^1 = a^2,\ a^2 \times a^2 = a^4,\ a^3 \times a^3 = a^6,\ \&c.$$

It follows then that *every factor, which is a square, must have an exponent which is an even quantity, and that the root of this factor is found by writing its letter with an exponent equal to half the original exponent.*

Thus we have

$$\sqrt{a^2} = a^1 \text{ or } a,\ \sqrt{a^4} = a^2,\ \sqrt{a^6} = a^3,\ \&c.$$

With respect to numerical factors, their roots are extracted when they admit of any, by the rules already given.

Whence the factors a^6, b^4, c^2, in the expression

$$\sqrt{64\, a^6\, b^4\, c^2},$$

are squares, and the number 64 is the square of 8 ; therefore, *as the expression proposed is the product of factors, which are squares, it will have for a root the product of the roots of these several factors* (CXXX) : and, consequently,

$$\sqrt{64\, a^6\, b^4\, c^2} = 8\, a^3\, b^2\, c.$$

CXXXI. In other cases, different from the above, *we must endeavour to resolve the proposed quantity. considered as a product, into two other products, one of which shall contain only such factors as are squares, and the other those factors which are not squares.* To effect this, we must consider each of the quantities separately.

Let there be, for example,

$$\sqrt{72\, a^4\, b^3\, c^5}.$$

We see that among the divisors of 72, the following are perfect squares, namely, 4, 9, and 36 ; if we take the greatest, we have

$$72 = 36 \times 2.$$

As the factor a^4 is a square, we separate it from the others; passing then to the factor b^3, which is not a square, since 3 is an odd number, we observe that this factor may be resolved into two others, b^2 and b, the first of which is a square ; we have then

$$b^3 = b^2.b\,;$$

it is obvious also that

$$c^5 = c^4.c.$$

By proceeding in the same manner with every letter, whose exponent is an odd number, the quantity is resolved thus,

$$72\, a^4 b^3 c^5 = 36.2\, a^4 b^2 . b\, c^4 . c\,;$$

by collecting the factors, which are squares, it becomes

$$36\, a^4 b^2 c^4 \times 2\, b\, c.$$

Lastly, taking the root of the first product and indicating that of the second, we have

$$\sqrt{72\, a^4 b^3 c^5} = 6\, a^2\, b\, c^2 \sqrt{2\, b\, c}.$$

See some examples of this kind of reduction, with the steps by which they are performed;

$$\sqrt{\frac{a^3}{b}} = \sqrt{a^2\, \frac{a}{b}} = a \sqrt{\frac{a}{b}} = a \sqrt{\frac{a\,b}{b^2}} = \frac{a}{b} \sqrt{a\,b}\,;$$

$$6 \sqrt{\frac{75}{98}\, a\, b^2} = 6 \sqrt{\frac{25\,.\,3\, a\, b^2}{49\,.\,2}} = 6 \sqrt{\frac{25\, b^2\,.\,3\, a}{49\,.\,2}} =$$

$$\frac{6\,.\,5}{7}\, b \sqrt{\frac{3\,a}{2}} = \frac{30\, b}{7} \sqrt{\frac{3\,a}{2}}\,;$$

$$\sqrt{\frac{a^2\, m^2}{n^2} + \frac{a^2\, m}{n}} = \sqrt{\frac{a^2\, m^2 + a^2\, m\, n}{n^2}} =$$

$$\sqrt{\frac{a^2}{n^2}\,(m^2 + m\, n)} = \frac{a}{n} \sqrt{m^2 + m\, n}.$$

It will be seen by the first of these examples, that the denominator of an algebraic fraction may be taken from under the radical sign by being made a complete square, in the same manner as we reduce the root of a numerical fraction (104).

CXXXII. We now proceed to the extraction of the square root of polynomials. It must here be recollected, that no binomial is a perfect square, because every simple quantity raised to a square produces only a simple quantity, and the square of a binomial always contains three parts (34).

It would be a great mistake to suppose the binomial $a + b$ to be the square root of $a^2 + b^2$, although, taken separately, a is the root of a^2, and b that of b^2: for the square of $a + b$, or $a^2 + 2\, a\, b + b^2$, contains the term $+ 2\, a\, b$, which is not found in the expression $a^2 + b^2$.

Let there be the trinomial

$$24\, a^2 b^3 c + 16\, a^4 c^2 + 9\, b^6.$$

In order to obtain from this expression the three parts, which compose the square of a binomial, we must arrange it with

reference to one of its letters, the letters a, for example; it then becomes

$$16\,a^4\,c^2 + 24\,a^2\,b^3\,c + 9\,b^6.$$

Now, whatever be the square root sought, if we suppose it arranged with reference to the same letter a, the square of its first term must necessarily form the first term $16\,a^4\,c^2$, of the proposed quantity; double the product of the first term of the root by the second must give the second term, $24\,a^2\,b^3\,c$, of the proposed quantity; and the square of the last term of the root must give exactly the last term, $9\,b^6$, of the proposed quantity. The operation may be exhibited, as follows:

$$\begin{array}{rl} 16\,a^4\,c^2 + 24\,a^2\,b^3\,c + 9\,b^6 & \left\{ \begin{array}{l} 4\,a^2\,c + 3\,b^3 \text{ root} \\ \overline{8\,a^2\,c + 3\,b^3} \end{array} \right. \\ -16\,a^4\,c^2 \qquad\qquad\qquad\qquad & \\ \hline +24\,a^2\,b^3\,c + 9\,b^6 & \\ -24\,a^2\,b^3\,c - 9\,b^6 & \\ \hline 0 \qquad\qquad 0 & \end{array}$$

We begin by finding the square root of the first term, $16\,a^4\,c^2$, and the result $4\,a^2\,c$ (130) is the first term of the root, which is to be written on the right, upon the same line with the quantity, whose root is to be extracted.

We subtract from the proposed quantity, the square $16\,a^4\,c^2$, of the first term, $4\,a^2\,c$, of the root; there remain then only the two terms $24\,a^2\,b^3\,c + 9\,b^6$.

As the term $24\,a^2\,b^3\,c$ is double the product of the first term of the root, $4\,a^2\,c$, by the second, we obtain this last, by dividing $24\,a^2\,b^3\,c$ by $8\,a^2\,c$, double of $4\,a^2\,c$, which is written below the root; the quotient $3\,b^3$ is the second term of the root.

The root is now determined; and, if it be exact, the square of the second term will be $9\,b^6$, or rather, double of the first term of the root $8\,a^2\,c$ together with the second $3\,b^3$, multiplied by the second, will reproduce the two last terms of the square (91); therefore we write $+\,3\,b^3$ by the side of $8\,a^2\,c$, and multiply $8\,a^2\,c + 3\,b^3$ by $3\,b^3$; after the product is subtracted from the two last terms of the quantity proposed, nothing remains; and we conclude, that this quantity is the square of $4\,a^2\,c + 3\,b^3$.

It is evident that the same reasoning and the same process may be applied to all quantities composed of three terms.

CXXXIII. When the quantity, whose root is to be extracted, has more than three terms, it is no longer the square of a binomial; but if we suppose it the square of a trinomial

$m + n + p$, and represent by l the sum $m + n$, this trinomial becoming now $l + p$, its square will be

$$l^2 + 2\,l\,p + p^2,$$

in which the square l^2 of the binomial $m + n$, produces, when developed, the terms $m^2 + 2\,m\,n + n^2$. Now, after we have arranged the proposed quantity, the first term will evidently be the square of the first term of the root, and the second will contain double the product of the first term of the root by the second of this root; we shall then obtain this last by dividing the second term of the proposed quantity by double the root of the first. Knowing then the two first terms of the root sought we complete the square of these two terms, represented here by l^2; subtracting this square from the proposed quantity, we have for a remainder

$$2\,l\,p + p^2,$$

a quantity, which contains double the product of l, or of the first binomial $m + n$, by the remainder of the root, plus the square of this remainder. It is evident, therefore, that we must proceed with this binomial as we have done with the first term m of the root.

$$\therefore\ (m + n + p)^2 = m^2 + n^2 + p^2 + 2\,m\,n + 2\,m\,p + 2\,n\,p.$$

Let there be, for example, the quantity

$64\,a^2\,b\,c + 25\,a^2\,b^2 - 40\,a^3\,b + 16\,a^4 + 64\,b^2\,c^2 - 80\,ab^2\,c$;

we arrange it with reference to the letter a, and make the same disposition of the several parts of the operation as in the above example.

$$\begin{array}{l|l}
16\,a^4 - 40\,a^3b + 25\,a^2b^2 - 80\,ab^2c + 64\,b^2c^2 & 4a^2 - 5ab + 8bc \\
\qquad\qquad\qquad + 64a^2bc & \hline 8a^2 - 5ab \\
-16\,a^4 & 8a^2 - 10ab + 8bc \\
\hline
\text{1st rem.} - 40a^3b + 25a^2b^2 - 80ab^2c + 64b^2c^2 & \\
\qquad\qquad\qquad + 64a^2bc & \\
\qquad + 40a^3b - 25a^2b^2 & \\
\hline
\text{2nd rem.}\ldots\ldots\ldots + 64a^2bc - 80ab^2c + 64b^2c^2 & \\
\qquad\qquad\qquad - 64a^2bc + 80ab^2c - 64b^2c^2 & \\
\hline
\qquad\qquad\qquad\quad 0 \qquad\quad 0 \qquad\quad 0 &
\end{array}$$

We extract the square root of the first term $16\,a^4$, and obtain $4a^2$ for the first term of the root sought, the square of which is to be subtracted from the proposed quantity.

We double the first term of the root, and write the result, $8a^2$, under the root; dividing by this the term $-40\,a^3b$, which begins the first remainder, we have $-5\,a\,b$ for the second term of the root; this is to be placed by the side of $8\,a^2$; we then

multiply the whole by this second term, and subtract the result from the remainder, upon which we are employed.

Thus we have subtracted from the proposed quantity the square of the binomial $4a^2 - 5ab$; the second remainder can contain only double the product of this binomial, by the third term of the root, together with the square of this term; we take then double the quantity $4a^2 - 5ab$, or

$$8a^2 - 10ab.$$

which is written under $8a^2 - 5ab$, and constitutes the divisor to be used with the second remainder; the first term of the quotient, which is $8bc$, is the third of the root.

This term we write by the side of $8a^2 - 10ab$, and multiply the whole expression by it; the product being subtracted from the remainder under consideration, nothing is left; the quantity proposed, therefore, is the square of

$$4a^2 - 5ab + 8bc.$$

The above operation, which is perfectly analogous to that, which has been already applied to numbers, may be extended to any length we please.

Miscellaneous Examples, in the Extraction of the Square Root, and in Reductions of Algebraic Expressions.

1. $$\sqrt{\left(\frac{a(\sqrt{3} + 6\sqrt{b} + 3b\sqrt{3}}{3\sqrt{b}}\right)} = \sqrt{\frac{a}{3}} - \sqrt{ab}$$

2. $$\sqrt{\left(a^2 + 6\frac{1}{9}a - 2\sqrt{a} - \frac{2a\sqrt{a}}{3} + 9\right)}$$

$$= a + 3 - \frac{\sqrt{a}}{3}$$

3. $$\sqrt{\left(9m^2 + 4n^2 + \frac{r^2}{9} + 2mr - 12mn - \frac{4}{3}nr\right)}$$

$$= 3m - 2n + \frac{r}{3}$$

4. $$\sqrt{\left(\frac{1}{4m^4n^4} - \frac{1}{n^2} - \frac{1}{m^2} + 2 + \frac{m^2}{n^2} + \frac{n^2}{m^2}\right)}$$

$$= \frac{1}{2mn} - \frac{m}{n} - \frac{n}{m}$$

5. $\sqrt{\left(\frac{3a-b}{2}-2\sqrt{\frac{a^2-b^2}{2}}\right)}=\sqrt{\frac{a+b}{2}}-\sqrt{a-b}$

6. $\sqrt{\left(\frac{a}{b}+3+\frac{b}{a}-2\sqrt{\frac{a}{b}}-2\sqrt{\frac{b}{a}}\right)}=\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}-1$

7. $\sqrt{\left(a+\frac{2}{\sqrt{a}}-2\sqrt{a}+1-\frac{2}{a}+\frac{1}{a^2}\right)}=\sqrt{a}-1+\frac{1}{a}$

8. $\sqrt{\left(\frac{m^2\ a^2}{n^4\ b^2}+2a+2b+\frac{m^2\ a}{n^2\ b}-\frac{m\ a}{n\ b}+\frac{n^2\ b^2}{m^2}-\frac{n\ b}{m}+\frac{1}{4}-\frac{m}{n}+\frac{m^2}{n^4}\right)}\quad\frac{m\ a}{n\ b}+\frac{n\ b}{m}-\frac{1}{2}+\frac{m}{n}$

9. $\sqrt{\left(\frac{r^4\ q^6}{9}-\frac{r^3\ q^4}{3}+r^2\ q^3+\frac{r^2\ q^2}{4}-\frac{3\ r\ q}{2}+\frac{9}{4}\right)}=\frac{r^2\ q^3}{3}-\frac{r\ q}{2}+\frac{3}{2}$

10. $\sqrt{\left[2x^2\,(2x^2+b^2)+\frac{2}{x^2}\left(b+\frac{2}{x^4}\right)+\frac{b^2}{4}+8\right]}=2x^2+\frac{b}{2}+\frac{2}{x^2}$

11. $\sqrt{\left[\frac{x^2\ y^2}{m^2}+\frac{m^2\ x^2}{y^2}+\frac{m^2\ y^2}{x^2}+2x^2+2y^2+2m^2\right]}=\frac{x\ y}{m}+\frac{m\ x}{y}+\frac{my}{x}$

12. $\sqrt{\left[\frac{a^2\ b^2}{3}\left(\frac{c}{3}-a\ c\right)-a^4\ b\left(\frac{2\ c}{3}+\frac{b}{4}+a+\frac{a^2}{b}\right)\right]}=\frac{a\ b\ c}{3}-\frac{a^2\ b}{2}-a^3$

13. $\sqrt{\left[\frac{1}{m\ n}\left(\frac{1}{m\ n}-\frac{2\sqrt{m}}{n}-\frac{2\sqrt{n}}{m}+\frac{m^2}{n}-2\sqrt{m\ n}+\frac{n^2}{m}\right)\right]}=\frac{1}{m\ n}+\frac{\sqrt{m}}{n}-\frac{\sqrt{n}}{m}$

14. $\sqrt{\left[a^2\left(\frac{a^2}{9}-\frac{10}{3\,b^2}\right)+b^2\left(2+\frac{9\,b^4}{a^6}\right)-\frac{30}{a^2}+\frac{25}{b^2}\right]}$
$=\frac{a^2}{3}+\frac{3\,b^2}{a^3}-\frac{5}{b^2}$

15. $\frac{a}{b}+\frac{c}{d}=\frac{a\,d+b\,c}{b\,d}$

16. $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}=\frac{a\,d\,f+b\,c\,f+b\,d\,e}{b\,d\,f}$

17. $\frac{3\,a}{5\,b}+\frac{c}{4\,d}+h=\frac{12\,a\,d+5\,b\,c+20\,b\,d\,h}{20\,b\,d}$

18. $\frac{a}{b}+\frac{c}{d}-\frac{e}{f}-\frac{g}{h}-k=$
$\frac{a\,d\,f\,h+b\,c\,f\,h-b\,d\,e\,h-b\,d\,f\,g-b\,d\,f\,h\,k}{b\,d\,f\,h}$

19. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}=\frac{b\,c-a\,c+a\,b}{a\,b\,c}$

20. $a-b-\frac{d}{ef}-\frac{c}{e\,g}=\frac{(a-b)\,e\,f\,g-d\,g-c\,f}{e\,f\,g}$

21. $\frac{a}{b^n}+\frac{c}{b^{n-r}}+\frac{d}{b^{n-2r}}=\frac{a+c\,b^r+d\,b^{2r}}{b^n}$

22. $\frac{a}{x^n}-\frac{c}{x^{n-r}}+\frac{d}{x^{n-r-s}}=\frac{a-c\,x+d\,x^{r+s}}{x^n}$

23. $c+2\,a\,b-3\,a\,c-\frac{b^2\,c-5\,a\,b^2\,c+a^3}{b^2-b\,c}=$
$\frac{2\,a\,b^3-b\,c^2+3\,a\,b\,c^2-a^3}{b^2-b\,c}$

24. $\frac{13\,a-5\,b}{4}-\frac{7\,a-2\,b}{6}-\frac{3\,a}{5}=\frac{89\,a-55\,b}{60}$

25. $\frac{3\,a-4\,b}{7}-\frac{2\,a-b-c}{3}+\frac{15\,a-4\,c}{12}=$
$\frac{85a-20b}{84}$

26. $\frac{3\,a+2\,b}{c}-\frac{5\,b\,d-2\,a-3\,d}{4\,c\,d}=\frac{12a\,d+3\,b\,d+2a+3d}{4\,c\,d}$

27. $\frac{a}{b}+\frac{a-3\,b}{c\,d}+\frac{a^2-b^2-a\,b}{b\,c\,d}=\frac{a\,c\,d-4\,b^2+a^2}{b\,c\,d}$

28. $\frac{(a+x)^{\frac{p}{q}-1}}{3b^2(c+x)^{\frac{m}{n}}} - \frac{b^{\frac{2}{3}}x^2(c+x)^{-\frac{m}{n}}}{(a+x)^{1-\frac{p}{2q}}} =$

$$\frac{(a+x)^{\frac{p}{2q}} - 3b^{\frac{2}{3}}x^2}{3b^2(c+x)^{\frac{m}{n}}(a+x)^{1-\frac{p}{2q}}}$$

29. $\frac{a}{a+2} + \frac{z}{a-z} = \frac{a^2+z^2}{a^2-z^2}$

30. $\frac{f+g}{3f-2g} - \frac{5f-2g}{2f-9g} = \frac{9fg - 13f^2 - 13g^2}{6f^2 - 31fg + 18g^2}$

31. $\frac{a}{b+x} - \frac{c}{x} + \frac{3c}{4x} + 2b = \frac{8bx^2 + (8b^2+4a-c)x - bc}{4bx + 4x^2}$

32. $\frac{3a+2x}{a+x} - \frac{5a-x}{a-x} + \frac{a}{2x} =$

$$\frac{a^3 - 4a^2x - 11ax^2 - 2x^3}{2x(a^2-x^2)}$$

33. $\frac{az}{a^2-z^2} - \frac{a-z}{a+z} = \frac{3az - a^2 - z^2}{a^2-z^2}$

34. $\frac{ac}{a^2-4y^2} + \frac{bd}{ac+2cy} = \frac{ac^2 + abd - 2bdy}{c(a^2-4y^2)}$

35. $\frac{a^3}{(a+b)^3} - \frac{ab}{(a+b)^2} + \frac{b}{a+b} = \frac{a^3 + ab^2 + b^3}{(a+b)^3}$

36. $\frac{a^m}{(a+b)^n} + \frac{a^{m-2}b^r}{(a+b)^{n-1}} - \frac{a^{m-3}b^r}{(a+b)^{n-2}}$

$$= \frac{a^m - a^{m-2}b^{r+1} - a^{m-1}b^{r+2}}{(a+b)^n}$$

37. $\frac{1}{1-z^2} - \frac{1}{m+1+(m-1)z^2} =$

$$\frac{m(1+z^2)}{m(1-z^4) + (1-z^2)^2}$$

38. $\frac{3}{4(1-x)^2} + \frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}$

$$= \frac{1+x+x^2}{1-x-x^4+x^5}$$

39. $\frac{3h}{(h-2x)^2}+\frac{2h+x}{(h+x)(h-2x)}-\frac{5}{h+x}=$
$\frac{20hx-22x^2}{(h+x)(h-2x)^2}$

40. $\frac{ax+x^2}{3bx-cx}=\frac{a+x}{3b-c}$

41. $\frac{14a^2-7ab}{10ac-5bc}=\frac{7a}{5c}$.

42. $\frac{12a^3x^4+2a^2x^5}{18ab^4x+3b^4x^2}=\frac{2a^2x^3}{3b^4}$

43. $\frac{6ac+9bc-5c^2}{12adf+18bdf-10cdf}=\frac{c}{2df}$

44. $\frac{45a^3b^4c+27a^6b^7cd-9a^4b^2d^3}{30a^2b^2c^3d^4+18a^7b^5c^3d^5-6a^3c^2d^7}=\frac{3ab^2}{2c^2d^4}$

45. $\frac{30a^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^4-4a^{-3}b^3d^{r+1}}=\frac{6a^{2n-1}bc^r}{4d^4}$

46. $\frac{5a^2+5ax}{a^2-x^2}=\frac{5a}{a-x}$

47. $\frac{a^3-x^3}{(a-x)^2}=\frac{a^2+ax+x^2}{a-x}$

48. $\frac{n^2-2n+1}{n^2-1}=\frac{n-1}{n+1}$

49. $\frac{a^2+(1+a)ay+y^2}{a^4-y^2}=\frac{a+y}{a^2-y}$

50. $\frac{ac+bd+ad+bc}{af+2bx+2ax+bf}=\frac{c+d}{f+2x}$

51. $\frac{6ac+10bc+9ad+15bd}{6c^2+9cd-2c-3d}=\frac{3a+5b}{3c-1}$

52. $\frac{n^3-2n^2}{n^2-4n+4}=\frac{n^2}{n-2}$

53. $\frac{x^2+2x-3}{x^2+5x+6}=\frac{x-1}{x+2}$

54. $\frac{9x^3+53x^2-9x-18}{x^2+11x+30}=\frac{9x^2-x-3}{x+5}$

55. $$\frac{(a+b)\,(a+b+c)\,(a+b-c)}{2\,a^2\,b^2+2\,a^2\,c^2+2\,b^2\,c^2-a^4-b^4-c^4}=$$

$$\frac{(a+b)\,(a+b+c)\,(a+b-c)}{4\,b^2\,c^2-(a^2-b^2-c^2)^2}=\frac{a+b}{(c+a-b)\,(b-a+c)}$$

56. $$\sqrt{a\,x}+\frac{a\,x}{a-\sqrt{a\,x}}=\frac{a\sqrt{a\,x}}{a-\sqrt{a\,x}}=\frac{a\sqrt{x}}{\sqrt{a}-\sqrt{x}}$$

57. $$\frac{a+\sqrt{-b}}{a-\sqrt{-b}}+\frac{a-\sqrt{-b}}{a+\sqrt{-b}}=\frac{2\,(a^2-b)}{a^2+b}$$

58. $$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=\frac{a+\sqrt{(a^2-x^2)}}{x}$$

59. $$\frac{\dfrac{a}{a-b}+\dfrac{b}{a+b}}{\dfrac{a}{a-b}-\dfrac{b}{a+b}}=\frac{a^2+2\,a\,b-b^2}{a^2+b^2}$$

60. $$\frac{1+\dfrac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{(a^2+x^2)}}$$

61. $$\frac{\sqrt{(1-x)}+\dfrac{1}{\sqrt{(1+x)}}}{1+\dfrac{1}{\sqrt{(1-x^2)}}}=\sqrt{(1-x)}$$

62. $$\frac{a^2+a\,x+x^2}{a^4+a^3\,x+a^2\,x^2+a\,x^3+x^4}=\frac{a^3-x^3}{a^5-x^5}$$

63. $$\frac{a^3-a^2\,x+a\,x^2-x^3}{a^5-a^4\,x+a^3\,x^2-a^2\,x^3+a\,x^4-x^5}=\frac{a^4-x^4}{a^6-x^6}$$

64. $$\frac{a^2+2\,a\,x+4\,x^2}{a^3-2\,a^2\,x+4\,a\,x^2-8\,x^3}=\frac{a^3+8\,x^3}{a^4-16\,x^4}$$

65. $$\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}=\frac{a-2b}{a+b-c}$$

Of the Formation of Powers and the Extraction of their Roots.

CXXXIV. The arithmetical operation, upon which the resolution of equations of the second degree depends, and by which we ascend from the square of a quantity to the quantity, from which it is derived, or to the square root, is only a particular case of a more general problem, namely, *to find a number, any power of which is known.* The investigation of this problem leads to a result, that is still termed a root, the different kinds being called degrees; but the process is to be understood only by a careful examination of the steps by which a power is obtained, one operation being the reverse of the other, as we observe with respect to division and multiplication, with which it will soon be perceived that this subject has other relations.

It is by multiplication, that we arrive at the powers of entire numbers (24), and it is evident, that those of fractions also are formed by raising the numerator and denominator to the power proposed (96).

So also the root of a fraction, of whatever degree, is obtained by taking the corresponding root of the numerator and that of the denominator.

As algebraic symbols are of great use in expressing every thing, which relates to the composition and decomposition of quantities, we shall first consider how the powers of algebraic expressions are formed, those of numbers being easily found by the methods that have already been given (24).

Table of the first Seven Powers of Numbers from 1 *to* 9.

1st	1	2	3	4	5	6	7	8	9
2d.	1	4	9	16	25	36	49	64	81
3rd	1	8	27	64	125	216	343	512	729
4th	1	16	81	256	625	1296	2401	4096	6561
5th	1	32	243	1024	3125	7776	16807	32768	59049
6th	1	64	729	4096	15625	46656	117649	262144	531441
7th	1	128	2187	16384	78125	279936	823543	2097152	4782969

This table is intended particularly to shew with what rapidity the higher powers of numbers increase, a circumstance

that will be found to be of great importance hereafter; we see, for instance, that the seventh power of 2 is 128, and that of 3 amounts to 4782969.

It will hence be readily perceived that the powers of fractions, properly so called, decrease very rapidly, since the powers of the denominator become greater and greater in comparison with those of the numerator. (See Note to page 100.) The seventh power of $\frac{1}{2}$, for example, is $\frac{1}{128}$, and that of $\frac{1}{3}$ is only

$$\frac{1}{4782969}.$$

CXXXV. It is evident from what has been said, that in a product each letter has for an exponent the sum of the exponents of its several factors (26), that *the power of a simple quantity is obtained by multiplying the exponent of each factor by the exponent of this power.*

The third power of $a^2 b^3 c$, for example, is found by multiplying the exponents 2, 3, and 1, of the letters a, b, and c, by 3, the exponent of the power required; we have then $a^6 b^9 c^3$; the operation may be thus represented,

$$a^2 b^3 c \times a^2 b^3 c \times a^2 b^3 c = a^{2.3} b^{3.3} c^{1.3}.$$

If the proposed quantity have a numerical coefficient, this coefficient must also be raised to the same power; thus the fourth power of $3 a b^4 c^5$, is

$$81 a^4 b^{16} c^{20}.$$

CXXXVI. With respect to the signs, with which simple quantities may be effected, it must be observed, that *every power, the exponent of which is an even number, has the sign +, and every power, the exponent of which is an odd number, has the same sign as the quantity from which it is formed.*

In fact powers of an even degree arise from the multiplication of an even number of factors; and the signs —, combined two and two in the multiplication, always give the sign + in the product (31). On the contrary, if the number of factors is uneven, the product will have the sign —, when the factors have this sign, since this product will arise from that of an even number of factors, multiplied by a negative factor.

CXXXVII. In order to ascend from the power of a quantity, to the root from which it is derived, we have only to re-

verse the rules given above, that is, *to divide the exponent of each letter by that, which marks the degree of the root required.*

Thus we find the *cube root*, or *the root of the third degree*, of the expression $a^6 b^9 c^3$, by dividing the exponents 6, 9, and 3, by 3, which gives

$$a^2 b^3 c.$$

When the proposed expression has a numerical coefficient, its root must be taken for the coefficient of the literal quantity, obtained by the preceding rule.

If it were required, for example, to find the fourth root of $81\, a^4 b^8 c^{20}$, we see, by referring to the table, art. 134, that 81 is the fourth power of 3; then, dividing the exponent of each of the letters by 4, we obtain for the result

$$3\, a\, b^2 c^5.$$

When the root of the numerical coefficient cannot be found by the table inserted above, it must be extracted by the methods to be given hereafter.

CXXXVIII. It is evident, that the roots of the literal part of simple quantities can be extracted, only when each of the exponents is divisible by that of the root; in the contrary case, we can only indicate the arithmetical operation, which is to be performed whenever numbers are substituted in the place of the letters.

We use for this purpose the sign $\sqrt{\ }$; but to designate the degree of the root, we place the exponent as in the following expressions,

$$\sqrt[3]{a},\ \sqrt[5]{a^4},$$

the first of which represents the cube root, or the root of the third degree of a, and the second the fifth root of a^4.

We may often simplify radical expressions of any degree whatever, by observing, according to art. 135, that *any power of a product is made up of the product of the same power of each of the factors*, and that, consequently, *any root of a product is made up of the product of the roots of the same degree of the several factors*. It follows from this last principle, that, *if the quantity placed under the radical sign have factors, which are exact powers of the degree denoted by this sign, the roots of these factors may be taken separately, and their product multiplied by the root of the other factors indicated by the sign.*

Let there be, for example,

$$\sqrt[5]{96\, a^5 b^7 c^{11}}.$$

It may be seen that,

$$96 = 32 \times 3 = 2^5 \,.\, 3,$$

that a^5 is the fifth power of a,

that $b^7 = b^5 . b^2$,

that $c^{11} = c^{10} \,.\, c$;

we have then

$$96\, a^5\, b^7\, c^{11} = 2^5\, a^5\, b^5\, c^{10} \times 3\, b^2\, c.$$

As the first factor, $2^5\, a^5\, b^5\, c^{10}$, has for its fifth root the quantity $2\, a\, b\, c^2$, the expression becomes

$$\sqrt[5]{96\, a^5\, b^7\, c^{11}} = 2\, a\, b\, c^2\, \sqrt[5]{3\, b^2\, c}.$$

CXXXIX. As every even power has the sign $+$ (136), a quantity, affected with the sign $-$, cannot be a power of a degree denoted by an even number, and it can have no root of this degree. It follows from this, *that every radical expression of a degree which is denoted by an even number, and which involves a negative quantity is imaginary*, thus

$$\sqrt[4]{-a},\ \sqrt[6]{-a^4},\ b + \sqrt[8]{-a\, b^7},$$

are imaginary expressions.

We cannot, therefore, either exactly or by approximation, assign for a degree, the exponent of which is an even number, any roots but those of positive quantities, and *these roots may be affected indifferently with the sign $+$, or $-$*, because, in either case, they will equally reproduce the proposed quantity with the sign $+$, and we do not know to which class they belong.

The same cannot be said of degrees expressed by an odd number, for here the powers have the same sign as their roots (136); and *we must give to the roots of these degrees the sign, with which the power is affected;* and no imaginary expressions occur.

CXL. It is proper to observe, that the application of the rule given in art. 137, for the extraction of the roots of simple quantities, by means of the exponent of their factors, leads to a more convenient method of indicating roots, which cannot be obtained algebraically, than by the sign $\sqrt{\ }$.

If it were required, for example, to find the third root of a^5, it is necessary according to the rule given above, to divide the exponent 5 by 3; but as we cannot perform the division, we have for the quotient the fractional number $\frac{5}{3}$; and this form of the exponent indicates, that the extraction of the root is not possible in the actual state of the quantity proposed. We may, therefore, consider the two expressions

$$\sqrt[3]{a^5} \text{ and } a^{\frac{5}{3}}$$

as equivalent.

The second. however, has this advantage over the first, that it leads, directly to a more simple form, which the quantity $\sqrt[3]{a^5}$ is capable of assuming; for if we take the whole number contained in the fraction $\frac{5}{3}$, we have $1 + \frac{2}{3}$ as an equivalent exponent; consequently,

$$a^{\frac{5}{3}} = a^{1+\frac{2}{3}} = a^{1} \times a^{\frac{2}{3}}$$

from which it is evident, that the quantity $a^{\frac{5}{3}}$ is composed of two factors, the first of which is rational, and the other becomes $\sqrt[3]{a^2}$.

The same result, indeed, may be obtained from the quantity under the form $\sqrt[3]{a^5}$, by the rule given in art. 138, but the fractional exponent suggests it immediately. We shall have occasion to notice in other operations the advantages of fractional exponents.

We will merely observe for the present, that as the division of exponents, when it can be performed, answers to the extraction of roots, the indication of this division under the form of a fraction is to be regarded as the symbol of the same operation; whence,

$$\sqrt[n]{a^m} \text{ and } a^{\frac{m}{n}}$$

are equivalent expressions.

We have rules then, which result from the assumed manner of expressing powers, which lead to particular symbols, as, in art. 37, we arrived at the expression $a^0 = 1$,

CXLI. It may be observed here, that as we divide one power by another. by subtracting the exponent of the latter from that of the former (36), fractions of a particular description may readily be reduced to new forms.

By applying the rule above referred to, we have

$$\frac{a^m}{a^n} = a^m$$

but if the exponent n of the denominator exceed the exponent m of the numerator, the exponent of the letter a in the second member will be negative.

If, for example, $m = 2$, $n = 3$, we have

but by another method of simplifying the fraction $\frac{a^2}{a^3}$ we find it is equal to $\frac{1}{a}$; the expressions

$$\frac{1}{a} \text{ and } a^{-1},$$

are therefore equivalent.

In general, we obtain by the rule for the exponents,

$$\frac{a^m}{a^{m+n}} = a^{m-m-n} = a^{-n},$$

and by another method

$$\frac{a^m}{a^{m+n}} = \frac{1}{a^n};$$

it follows from this, that the expressions

$$\frac{1}{a^n} \text{ and } a^{-n},$$

are equivalent.

In fact, the sign —, which precedes the exponent n, being taken in the sense defined in art. 62, shews that the exponent in question arises from a fraction, the denominator of which contains the factor a, n times more than the numerator, which fraction is indeed $\frac{1}{a^n}$; we may, therefore, in any case which occurs, substitute one of these expressions for the other.

The quantity $\frac{a^2\, b^5}{c^2\, d^3}$, for example, being considered as equivalent to

$$a^2\, b^5 \times \frac{1}{c^2} \times \frac{1}{d^3},$$

may be reduced to the following form,

$$a^2\, b^5 c^{-2} d^{-3};$$

that is, *we may transfer to the numerator all the factors of the denominator, which have positive exponents, by giving to their exponents the sign —.*

Reciprocally, *when a quantity contains factors, which have negative exponents, we may convert them into a denominator, observing merely to give to their exponents the sign +*; thus the quantity

$$a^2\, b^5\, c^{-2}\, d^{-3},$$

becomes

$$\frac{a^2\, b^5}{c^2\, d^3}.$$

Of the Formation of the Powers of Compound Quantities.

CXLII. We shall begin this section by observing, that the powers of compound quantities are denoted by including these quantities in a parenthesis, to which is annexed the exponent of the power

The expression

$$(4a^2 - 2ab + 5b^2)^3,$$

for example, denotes the third power of the quantity

$$4a^4 - 2ab + 5b^4.$$

This power may also be expressed thus,

$$\overline{4a^4 - 2ab + 5b^4}^{\,3}.$$

CXLIII. Binomials next to simple quantities are the least complicated, yet if we undertake to form powers of these by successive multiplications, we in this way arrive only at particular results, as in art. 34, we obtain the second and third power; thus

$$\begin{aligned}
(x+a)^2 &= x^2 + 2a + a^2,\\
(x+a)^3 &= x^3 + 3ax^2 + 3a^2x + a^3,\\
(x+a)^4 &= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4,
\end{aligned}$$

&c.

It is not easy from this table to fix upon the law, which determines the value of the numerical coefficients. But by considering how the terms are multiplied into each other, we perceive, that the coefficients have their origin in reductions depending on the equality of the factors, which form a power. This is rendered very evident by an arrangement, which prevents these reductions from taking place.

It is sufficient for this purpose, to give to the several binomials to be multiplied, different second terms. If we take, for example,

$$x + a,\ x + b,\ x + c,\ x + d,\ \&c.$$

by performing the multiplications indicated below, and placing in the same column the terms, which involve the same power of x, we shall immediately find, that

$$\begin{array}{rlllll}
(x+a)(x+b) = & x^2 & + ax & + ab & & \\
 & & + bx & & & \\
(x+a)(x+b)(x+c) = & x^3 & + ax^2 & + abx & + abc & \\
 & & + bx^2 & + acx & & \\
 & & + cx^2 & + bcx & & \\
(x+a)(x+b)(x+c)(x+d) = & x^4 & + ax^3 & + abx^2 & + abcx & + abcd \\
 & & + bx^3 & + acx^2 & + abdx & \\
 & & + cx^3 & + adx^2 & + acdx & \\
 & & + dx^3 & + bcx^2 & + bcdx & \\
 & & & + bdx^2 & & \\
 & & & + cdx^2. & &
\end{array}$$

Without carrying these products any further, we may discover the law according to which they are formed.

By supposing all the terms involving the same power of x, and placed in the same column, to form only one, as, for example,

$$a\,x^3 + b\,x^3 + c\,x^3 + d\,x^3 = (a + b + c + d)\,x^3,$$
&c.

(1.) *We find in each product one term more than there are units in the number of factors.*

(2.) *The exponent of* x *in the first term is the same as the number of factors, and goes on decreasing by unity in each of the following terms.*

(3.) *The greatest power of* x *has unity for its coefficient; the following, or that, whose exponent is one less, is multiplied by the sum of the second terms of the binomials; that, whose exponent is two less, is multiplied by the sum of the different products of the second terms of the binomials taken two and two; that, whose exponent is three less, is multiplied by the sum of the different products of the second term of the binomials, taken three and three, and so on; in the last term, the exponent of* x, *being considered as zero* (37), *is equal to that of the first diminished by as many units as there are factors employed, and this term contains the product of all the second terms of the binomials.*

It is manifest, that the form of these products must be subject to the same laws, whatever be the number of factors; as may be shewn by other evidence beside that from analogy.

CXLIV. It will be seen immediately, that the products, of which we are speaking, must contain the successive powers of x, from that, whose exponent is equal to the number of factors employed, to that, whose exponent is zero. To present this proposition under a general form, we shall express the number of factors by the letter m; the successive powers of x will then be denoted by

$$x^m,\ x^{m-1},\ x^{m-2},\ \&c.$$

We shall employ the letters, A, B, C, ... Y, to express the quantities, by which these powers, beginning with x^{m-1}, are to be multiplied; but as the number of terms, which depends on the particular value given to the exponent, will remain indeterminate, so long as this exponent has no particular value, we can write only the first and last terms of the expression, designating the intermediate terms by a series of points.

The formula then

$$x^m + A\,x^{m-1} + B\,x^{m-2} + C\,x^{m-3} \ldots\ldots + Y,$$

represents the product of any number m of factors,

$$x + a,\ x + b,\ x + c,\ x + d,\ \&c.$$

If we multiply this by a new factor $x + l$, it becomes

$$\left.\begin{array}{llllll} x^{m+1} & + A\,x^m & + B\,x^{m-1} & + C\,a^{m-} & & \\ & + l\,x^m & + l\,A\,x^{m-1} & + l\,B\,x^{m} & & + l\,Y \end{array}\right\}.$$

It is evident, 1. that if A is the sum of the m second terms a, b, c, d, &c. $A + l$ will be that of the $m + 1$ second terms a, b, c, d, &c. l, and that consequently the expression employed to denote the coefficient will be true for the product of the degree $m + 1$, if it is true for that of the degree m.

2. If B is the sum of the products of the m quantities a, b, c, d, &c. taken two and two, $B + l\,A$ will express that of the products of the $m + 1$ quantities a, b, c, d, &c. l, taken also two and two; for A being the sum of the first, $l\,A$ will be that of their products by the new quantity introduced l; therefore the expression employed will be true for the degree $m + 1$, if it is for the degree m.

3. If C is the sum of the products of the m quantities a, b, c, d, &c. taken three and three, $C + l\,B$ will be that of the products of the $m + 1$ quantities a, b, c, d, &c. l, taken also three and three, since $l\,B$, from what has been said, will express the sum of the products of the first taken two and two, multiplied by the new quantity introduced l; therefore, the expression employed will be true for the degree $m + 1$, if it is true for the degree m.

It will be seen, that this mode of reasoning may be extended to all the terms, and that the last, $l\,Y$ will be the product of $m + 1$ second terms.

The propositions laid down in art. 143, being true for expressions of the fourth degree, for example, will be so, according to what has just been proved, for those of the fifth, for those of the sixth, and, being extended thus from one degree to another, they may be shown to be true generally.

It follows from this, that the product of any number whatever m, of binomial factors $x + a$, $x + b$, $x + c$, $x + d$, &c. being represented by

$$x^m + A\,x^{m-1} + B\,x^{m-2} + C\,x^{m-3} + \&c.$$

A will always be the sum of the m letters a, b, c, &c., B that of the products of these quantities, taken two and two, C, that of the products of the quantities, taken three and three, and so on.

To comprehend the law of this expression in a single term, we take one, whose place is determinate, and which may be represented by $N\,x^{m-n}$.

This term will be the second, if we make $n = 1$, the third, if we make $n = 2$, the eleventh, if we make $n = 10$, &c. In the first case, the letter N will be the sum of the m letters, a, b, c, &c., in the second, that of their products, when taken two and two; in the third, that of their products, when taken ten and ten; and in general, that of their products, taken n and n.

CXLV. To change the products

$$(x + a)\ (x + b),\ (x + a)\ (x + b)\ (x + c),$$
$$(x + a)\ (x + b)\ (x + c)\ (x + d),\ \&c.$$

into powers of $x + a$, namely, into

$$(x + a)^2 \qquad (x + a)^3,$$
$$(x + a)^4, \qquad \&c.$$

it is only necessary to make, in the development of these products,

$$a = b, \qquad a = b = c,$$
$$a = b = c = d, \qquad \&c.$$

All the quantities, by which the same power of x is multiplied, become in this case equal; thus the coefficient of the second term, which in the product

$$(x + a)\ (x + b)\ (x + c)\ (x + d) \text{ is } a + b + c + d,$$

is changed into $4\,a$; that of the third term in the same product, which is,

$$a\,b + a\,c + a\,d + b\,c + b\,d + c\,d,$$

becomes $6\,a^2$. Hence it is easy to see, that the coefficients of the different powers of x will be changed into a single power of a, repeated as many times as there are terms, and distinguished by the number of factors contained in each of these terms. Thus, the coefficient N, by which the power x^{m-n} is multiplied, will, in the general development, be that power of a denoted by n, or a^n, repeated as many times, as we can form different products by taking in every possible way a number n of letters from among a number m; to find the coefficient of the term containing x^{m-n} then is reduced to finding the number of these products.

CXLVI. In order to perform the problem just mentioned, it is necessary to distinguish arrangements or *permutations* from products or *combinations*. Two letters, a and b, give only one product, but admit of two arrangements, $a\,b$ and $b\,a$; three

letters, a, b, c, which give only one product, admit of six arrangements (88), and so on.

To take a particular case, we will suppose the whole number of letters to be nine, namely,

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,$$

and that it is required to arrange them in sets of seven. It is evident, that if we take any arrangement we please, of six of these letters $a\ b\ c\ d\ e\ f$, for example, we may join successively to it each of the three remaining letters, g, h, and i; we shall then have three arrangements of seven letters, namely,

$$a\,b\,c\,d\,e\,f\,g, \qquad a\,b\,c\,d\,e\,f\,h, \qquad a\,b\,c\,d\,e\,f\,i.$$

What has been said of a particular arrangement of six letters, is equally true of all; we conclude, therefore, that each arrangement of six letters will give three of seven, that is, as many as there remain letters, which are not employed. If, therefore, the number of arrangements of six letters be represented by P, we shall obtain the number consisting of seven letters by multiplying P by 3 or $9 - 6$. Representing the numbers 9 and 7 by m and n, and regarding P as expressing the number of arrangements, which can be furnished by m letters, taken $n - 1$ at a time, the same reasoning may be employed: we shall thus have for the number of arrangements of n letters,

$$P\,[m - (n - 1)], \qquad \text{or} \qquad P\,(m - n + 1).$$

This formula comprehends all the particular cases, that can occur in any question. To find, for example, the number of arrangements, that can be formed out of m letters, taken two and two, or two at a time, we make $n = 2$, which gives

$$n - 1 = 1\,;$$

we have then

$$P = m\,;$$

for P will in this case be equal to the number of letters taken one at a time; there results then from this

$$m\,(m - 2 + 1), \qquad \text{or} \qquad m\,(m - 1),$$

for the number of arrangements taken two and two.

Again, taking

$$P = m\,(m - 1) \qquad \text{and} \qquad n = 3,$$

we find for the number of arrangements, which m letters admit of, taken three and three,

$$m\,(m - 1)\,(m - 3 + 1) = m\,(m - 1)\,(m - 2).$$

Making

$$P = m\,(m - 1)\,(m - 2) \quad \text{and } n = 4,$$

we obtain

$$m(m-1)(m-2)(m-3)$$

for the number of arrangements taken four and four. We may thus determine the number of arrangements, which may be formed from any number whatever of letters*.

CXLVII. Passing now from the number of arrangements of n letters, to that of their different products, we must find the number of arrangements, which the same product admits of. In order to this, it may be observed, that if in any of these arrangements, we put one of the letters in the first place, we may form of all the others as many permutations, as the product of $n-1$ letters admits of. Let us take, for example, the product $a\,b\,c\,d\,e\,f\,g$, composed of seven letters; we may, by putting a in the first place, write this product in as many ways, as there are arrangements in the product of six letters $b\,c\,d\,e\,f\,g$; but each letter of the proposed product may be placed first. Designating then the number of arrangements, of which a product of six letters is susceptible, by Q, we shall have $Q \times 7$ for that of the arrangements of a product of seven letters. It follows from this, that if Q designate the number of arrangements, which may be formed from a product of $n-1$ letters, $Q\,n$ will express the number of arrangements of a product of n letters.

Any particular case is readily reduced to this formula; for making $n=2$, and observing, that when there is only one letter, $Q=1$, we have $1 \times 2 = 2$ for the number of arrangements of a product of two letters. Again, taking $Q = 1 \times 2$ and $n=3$, we have $1 \times 2 \times 3 = 6$ for the number of arrangements of a product of three letters; further, making $Q = 1 \times 2 \times 3$ and $n=4$, there result $1 \times 2 \times 3 \times 4$, or 24 possible arrangements in a product of four letters, and so on.

CXLVIII. What we have now said being well understood, it will be perceived, that by dividing the whole number of arrange-

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* In these arrangements, it is supposed by the nature of the inquiry, that there are no repetitions of the same letter; but the theory of permutations and combinations, which is the foundation of the doctrine of chances, embraces questions in which they occur. The effect may be seen in the example we have selected, by observing, that we may write indifferently each of the 9 letters $a, b, c, d, e, f, g, h, i$, after the product of 6 letters a, b, c, d, e, f. Designating, therefore, the number of arrangements, taken six at a time, by P, we shall have $P' \times 9$ for the number of arrangements taken 7 at a time. For the same reason, if P denote the number of arrangements of m letters, taken $n-1$ at a time, that of their arrangements, when taken n at a time, will be $P\,m$.

This being admitted, as the number of arrangements of m letters, taken one at a time, is evidently m, the number of arrangements, when taken 2 and 2, will be $m \times m$, or m^2, when taken 3 and 3, the number will be $m \times m \times m$, or m^3; and lastly, m^n will express the number of arrangements, when they are taken n and n.

ments obtained from m letters, taken n at a time, by the number of arrangements of which the same product is susceptible, we have for a quotient the number of the different products, which are formed by taking in all possible ways n factors among these m letters. This number will, therefore, be expressed by $\frac{P\,(m-n+1)}{Q\,n}$ *; which being considered in connection with what was laid down in art. 145., will give $\frac{P\,(m-n+1)}{Q\,n}\,a^n\,x^{m-n}$ for the term containing x^{m-n} in the development of $(x+a)^m$.

It is evident, that the term which precedes this will be expressed by $\frac{P}{Q}\,a^{n-1}\,x^{m-n+1}$; for in going back towards the first term, the exponent of x is increased by unity, and that of a diminished by unity; moreover, P and Q are the quantities, which belong to the number $n-1$.

CXLIX. If we make $\frac{P}{Q}=M$, the two successive terms indicated above become

$$M\,a^{n-1}\,x^{m-n+1} \text{ and } M\,\frac{(m-n+1)}{n}\,a^n\,x^{m-n}.$$

These results show how each term in the development of $(x+a)^m$, is formed from the preceding.

Setting out from the first term, which is x^m, we arrive at the second, by making $n=1$; we have $M=1$, since x^m has only unity for its coefficient; the result then is $\frac{1\times m}{1}\,a\,x^{m-1}$, or $\frac{m}{1}\,a\,x^{m-1}$. In order to pass to the third term, we make $M=\frac{m}{1}$, and $n=2$, and we obtain $\frac{m\,(m-1)}{1\,.\,2}\,a^2\,x^{m-2}$. The fourth is

* It may be observed, that if we make successively

$$n=2, \qquad n=3, \qquad n=4, \text{ \&c.}$$

the formula $\frac{P\,(m-n+1)}{Q\,n}$ becomes

$$\frac{m\,(m-1)}{1\,.\,2},\quad \frac{m\,(m-1)\,(m-2)}{1\,.\,2\,.\,3},\quad \frac{m\,(m-1)\,(m-2)\,(m-3)}{1\,.\,2\,.\,3\,.\,4}, \text{ \&c.}$$

numbers, which express respectively, how many combinations may be made of any number m of things, taken two and two, three and three, four and four, &c.

found by supposing $M = \frac{m(m-1)}{1 \cdot 2}$, and $n = 3$, which gives $\frac{m(m-1)}{1 \cdot 2 \cdot 3}\frac{(m-2)}{} a^3 x^{m-3}$, and so on; whence we have the formula

$$(x + a)^m = x^m + \frac{m}{1} a x^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2}$$
$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^3 x^{m-3} + \&c.$$

which may be converted into this rule.

To pass from one term to the following, we multiply the numerical coefficients by the exponent of x *in the first, divide by the number which marks the place of this term, increase by unity the exponent of* a, *and diminish by unity the exponent of* x.

Although we cannot determine the number of terms of this formula without assigning a particular value to m; yet, if we observe the dependence of the terms upon each other, we can have no doubt respecting the laws of their formation, to whatever extent the series may be carried. It will be seen, that

$$\frac{m(m-1)(m-2) \ldots\ldots (m-n+1)}{1 \;.\; 2 \;.\; 3 \ldots\ldots \qquad n} a^n x^{m-n}$$

expresses the term, which has n terms before it.

This last formula is called the *general term* of the series

$$x^m + \frac{m}{1} a x^{m-1} + \frac{m(m-1)}{1 \cdot 2} a^2 x^{m-2} + \&c.$$

because if we make successively

$$n = 1,\ n = 2,\ n = 3, \&c.$$

it gives all the terms of this series.

CL. Now, if $(x + a)^5$ be developed, according to the rule given in the preceding article; the first term being

$$x^5 \text{ or } a^0 x^5 \text{ (37)},$$

the second will be

$$\frac{5}{1} a^1 x^4 \quad \text{or} \quad 5\, a\, x^4,$$

the third

$$\frac{5 \times 4}{2} a^2 x^3 \text{ or } \quad 10 \quad a^2 x^3,$$

the fourth

$$\frac{10 \times 3}{3} a^3 x^2 \quad \text{or} \quad 10 a^3 x^2$$

the fifth

$$\frac{10 \times 2}{4} a^4 x \quad \text{or} \quad 5 a^4 x,$$

the sixth

$$\frac{5 \times 1}{5} a^5 x^0 \quad \text{or } a^5.$$

Here the process terminates, because in passing to the following term it would be necessary to multiply by the exponent of x in the sixth, which is zero.

This may be shewn by the formula; for the seventh term, having for a numerical coefficient

$$\frac{m\,(m-1)\,(m-2)\,(m-3)\,(m-4)\,(m-5)}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}$$

contains the factor $m - 5$, which becomes $5 - 5 = 0$; and this same factor entering into each of the subsequent terms, reduces it to nothing.

Uniting the terms obtained above, we have

$$(x + a)^5 = x^5 + 5\,a\,x^4 + 10\,a^2\,x^3 + 10\,a^3\,x^2 + 5\,a^4\,x + a^5.$$

CLI. Any power whatever of any binomial may be developed by the formula given in art. 149. If it were required, for example, to form the sixth power of $2\,x^3 - 5\,a^3$, we have only to substitute in the formula the powers of $2\,x^3$ and $-5\,a^3$ respectively for those of x and a; since if we make

$$2\,x^3 = x' \text{ and } -5\,a^3 = a',$$

we have

$$(2\,x^3 - 5\,a^3)^6 = (x' + a')^6 =$$
$$x'^6 + 6\,a'\,x'^5 + 15\,a'^2\,x'^4 + 20\,a'^3\,x'^3$$
$$+ 15\,a'^4\,x'^2 + 6\,a'^5\,x' + a'^6 \;(149),$$

and it is only necessary to substitute for x' and a' the quantities, which these letters designate. We have then

$$(2\,x^3)^6 + 6\,(-5\,a^3)\,(2\,x^3)^5 + 15\,(-5\,a^3)^2\,(2\,x^3)^4$$
$$+ 20\,(-5\,a^3)^3\,(2\,x^3)^3 + 15\,(-5\,a^3)^4\,(2\,x^3)^2$$
$$+ 6\,(-5\,a^3)^5\,(2\,x^3) + (-5\,a^3)^6,$$

or

$$64\,x^{18} - 960\,a^3\,x^{15} + 6000\,a^6\,x^{12}$$
$$- 20000\,a^9\,x^9 + 37500\,a^{12}\,x^6$$
$$- 37500\,a^{15}\,x^3 + 15625\,a^{18}.$$

The terms produced by this development are alternately positive and negative; and it is manifest, that they will always be so, when the second term of the proposed binomial has the sign —.

CLII. The formula given in art. 149, may be so expressed as to facilitate the application of it in cases analogous to the preceding.

Since

$$x^{m-1} = \frac{x^m}{x}, \quad x^{m-2} = \frac{x^m}{x^2}, \quad x^{m-3} = \frac{x^m}{x^3}, \text{ \&c.}$$

the formula may be written

$$x^m + \frac{m}{1}\frac{a}{x}x^m + \frac{m\ (m-1)}{1\ .\ 2}\frac{a^2}{x^2}x^m + \text{\&c.}$$

which may be reduced to

$$x^m\left\{1 + \frac{m}{1}\frac{a}{x} + \frac{m\ (m-1)}{1\ .\ 2}\frac{a^2}{x^2} + \frac{m\ (m-1)\ (m-2)}{1\ .\ 2\ .\ 3}\frac{a^3}{x^3} + \text{\&c}\right\}$$

by insulating the common factor x^m. In applying this formula, the several steps are, *to form the series of numbers,*

$$\frac{m}{1}, \quad \frac{m-1}{2}, \quad \frac{m-2}{3}, \quad \frac{m-3}{4}, \text{ \&c.}$$

to multiply the first by the fraction $\frac{a}{x}$, *then this product by the second and also by the fraction* $\frac{a}{x}$, *then again this last result by the third and by the fraction* $\frac{a}{x}$, *and so on; to unite all these terms, and add unity to the sum; and lastly, to multiply the whole by the factor* x^m.

In the example $(2\ x^3 - 5\ a^3)^6$, we must write $(2\ x^3)^6$ in the place of x^m, and $-\frac{5\ a^3}{2\ x^3}$ in that of $\frac{a}{x}$. We shall leave the application of the formula as an exercise for the learner*.

---

* The formula for the development of $(x + a)^m$ answers for all values of the exponent m, and is equally applicable to cases in which the exponent is fractional or negative This property, which is very important, is generally demonstrated by the Differential and Integral Calculus.

For example

$$\left(1 \pm x\right)^m = 1 \pm \frac{m}{1}x + \frac{m}{1}\frac{m-1}{2}x^2 \pm \frac{m.}{1}\frac{m-1.}{2}\frac{m-2.}{3}x^3 + \frac{m.}{1}\frac{m-1.}{2}\frac{m-2.}{3}\frac{m-3.}{4} \pm \ldots\ldots \frac{m.\ m-1 \ldots\ldots (m-n+1)}{1\quad 2 \qquad\qquad n}x^{m-n}$$

$\pm$, &c.

CLIII. We may easily reduce the development of the power of the polynomial whatever, to that of the powers of a binomial, as may be shown with respect to the trinomial $a + b + c$, the third power for instance being required.

First, we make $b + c = m$, we then obtain

$$(a + b + c)^3 = (a + m)^3 = a^3 + 3\,a^2\,m + 3\,a\,m^2 + m^3;$$

substituting for m the binomial $b + c$, which it represents, we have

$$(a + b + c)^3 = a^3 + 3\,a^2\,(b + c) + 3\,a\,(b + c)^2 + (b + c)^3.$$

It only remains for us to develop the powers of the binomial $b + c$, and to perform the multiplications, which are indicated; we have then

$$\begin{array}{llll} a^3 + 3\,a^2\,b & + 3\,a\,b^2 & + & b^3 \\ \quad + 3\,a^2\,c & + 6\,a\,b\,c & + 3\,b^2 & c \\ & + 3\,a\,c^2 & + 3\,b & c^2 \\ & & + & c^3. \end{array}$$

Of the Extraction of the Roots of Compound Quantities.

CLIV. Having explained the formation of the powers of compound quantities, we now pass to the extraction of their roots, beginning with the cube root of numbers.

In order to extract the cube root of numbers, we must first become acquainted with the cubes of numbers, consisting of only one figure; these are given in the second line of the following table;

1	2	3	4	5	6	7	8	9
1	8	27	64	125	216	343	512	729

and the cube of 10 being 1000, no number consisting of three figures can contain the cube of a number consisting of more than one.

The cube of a number consisting of two figures is formed in a manner analogous to that, by which we arrive at the square; for if we resolve this number into tens and units, designating the first by a, and the second by b, we have

$$(a + b)^3 = a^3 + 3\,a^2\,b + 3\,a\,b^2 + b^3.$$

Hence it is evident, that *the cube, or third power, of a number composed of tens and units, contains four parts, namely, the cube of the tens, three times the square of the tens multiplied by the units, three times the tens multiplied by the square of the units, and the cube of the units.*

If it were required to find the third power of 47, by making $a = 4$ tens or 40, $b = 7$ units, we have

$$\begin{aligned} a^3 &= 64000 \\ 3\,a^2\,b &= 33600 \\ 3\,a\,b^2 &= 5880 \\ b^3 &= 343 \end{aligned}$$

Total, $103823 = 47 \times 47 \times 47$.

Now to go back from the cube 103823 to its root 47, we begin by observing that 64000, the cube of the 4 tens, contains no significant figure inferior to thousands; in seeking the cube of the tens therefore, we may neglect the hundreds, the tens, and the units of the number 103823. Pursuing, therefore, a method similar to that employed in extracting the square root, we separate, by a comma, the first three figures on the right; the greatest cube contained in 103 will be the cube of the tens. It is evident from the table, that this cube is 64, the root of which is 4; we therefore put 4 in the place assigned for the root. We then subtract 64 from 103; and by the side of the remainder, 39, bring down the last three figures.

103,823	47
64	48
398,23	

The whole remainder, 39823, contains still three parts of the cube, namely, three times the square of the tens multiplied by the units, or $3\,a^2\,b$, three times the tens multiplied by the square of the units, or $3\,a\,b^2$, and the cube of the units, or b^3. If the value of the product $3\,a^2\,b$ were known, we might obtain the units b, by dividing this product by $3\,a^2$, which is a known quantity, the tens being now found; but on the supposition that the product, $3\,a^2\,b$, is unknown, we readily perceive, that it can have no figure inferior to hundreds, since it contains the factor a^2, which represents the square of the tens; it must, therefore, be found in the part 398, which remains on the left of the number 39823, after the tens and units have been separated, and which contains, besides this product, the hundreds arising from the product, $3\,a\,b^2$, of the tens by the square of the units, and from the cube b^3, of the units.

If we divide 398 by 48, which is triple the square of the tens, $3\,a^2$ or 3×16, we obtain 8 for the quotient; but from what precedes, it appears that we ought not to adopt this figure for the units of the root sought, until we have made trial of it, by employing it in forming the last three parts of the cube, which must be contained in the remainder 39823. Making $b = 8$, we find

$$\begin{aligned} 3\,a^2\,b &= 38400 \\ 3\,a\,b^2 &= 7680 \\ b^3 &= 512 \end{aligned}$$

Total, 46592

As this result exceeds 39823, it is evident that the number 8 is too great for the units of the root. If we make a similar trial with 7, we find that it answers to the above conditions; 47 therefore is the root sought.

Instead of verifying the last figure of the root in the manner above described, we may raise the whole number expressed by the two figures, immediately to a cube; and this last method is generally preferred to the other. Taking the number 48 and proceeding thus, we find

$$48 \times 48 \times 48 = 110592.$$

As the result is greater than the proposed number, it is evident, that the figure 8 is too large.

CLV. What we have laid down in the above example may be applied to all cases, where the proposed number consists of more than three figures and less than seven. Having separated the first three figures on the right, we seek the greatest cube in the part which remains on the left, and write its root in the usual place; we subtract this cube from the number to which it relates, and to the remainder bring down the last three figures; separating now the tens and the units, we proceed to divide what remains on the left, by three times the square of the tens found; but before writing down the quotient as a part of the root, we verify it by raising to the cube the number consisting of the tens known, together with this figure under trial. If the result of this operation is too great, the figure for the units is to be diminished; we then proceed in the same manner with a less figure, and so on, until a root is found, the cube of which is equal to the proposed number, or is the greatest contained in this number, if it does not admit of an exact root. As we have often remainders that are very considerable, we will here add to what has been said, a method, by which it may be soon discovered, whether or not the unit figure of the root be too small.

The cube of $a + b$, when $b = 1$, becomes that of $a + 1$,

or $$a^3 + 3a^2 + 3a + 1,$$

a quantity, which exceeds a^3, the cube of a, by

$$3a^2 + 3a + 1.$$

Hence it follows, that *whenever the remainder, after the cube root has been extracted, is less than three times the square of the root, plus three times the root, plus unity, this root is not too small.*

CLVI. In order to extract the root of 105823817, it may be observed, that whatever be the number of figures in this root, if we resolve it into units and tens, the cube of the tens cannot enter into the last three figures on the right, and must consequently be found in 105823. But the greatest cube contained in 105823 must have more than one figure for its root; this root then may be resolved into units and tens, and. as the cube of the tens has no figure inferior to thousands, it cannot enter into the three last figures 823. If, after these are separated, there remain more than three figures on the left, we may repeat the reasoning just employed, and thus, dividing the number proposed into portions of three figures each, proceeding from right to left, and observing that the last portion may contain less than three figures, we come at length to the place occupied by the cube of the units of the highest order in the root sought.

Having thus taken the preparatory steps, we seek, by the rule given in the preceding article, the cube root of the two first portions on the left, and find for the result 47; we subtract the cube of this number from the two first portions, and to the remainder 2000 bring down the following portion 817. The number 2000817 will then contain the last three parts of the cube of a number, the tens of which are 47, and the units remain to be found. These units are therefore obtained as in the example given in the preceding article, by separating the last two figures on the right of the remainder, and dividing the part on the left by 6627, triple the square of 47. Then making trial with the quotient 3, arising from this division, by raising 473 to a cube, we obtain for the result the proposed number, since this number is a perfect cube.

105,823,8 17	473
64	48
41 8,23	6627
103 8 23	
2 0 00 8,17	
105 8 23 8 17	
000 0 00 0 00	

The explanation, we have given, of the above example, may take the place of a general rule. If the number proposed had contained another portion, we should have continued the operation, as we have done for the third; and it is to be recollected always, that a cipher must be placed in the root, if the number to be divided on the left of the remainder happen not to contain the number used as a divisor; we should then bring down the following portion, and proceed with it as with the preceding.

CLVII. *Since the cube of a fraction is found by multiplying this fraction by its square, or which amounts to the same thing, by taking the cube of the numerator and that of the*

denominator ; reversing this process, we arrive at the root, by extracting the root of the new numerator and that of the new denominator. The cube of $\frac{5}{6}$, for example, is $\frac{125}{216}$; taking the cube root of 125 and of 216, we find $\frac{5}{6}$.

We always proceed in this way, when the numerator and denominator are perfect cubes; but when this is not the case, we may avoid the necessity of extracting the root of the denominator, by multiplying the two terms of the proposed fraction by the square of this denominator. The denominator thence arising, will be the cube of the original denominator; and it will be only necessary then to find the root of the numerator. If we have, for example, $\frac{3}{5}$, by multiplying the two terms of this fraction by 25, the square of the denominator, we obtain

$$\frac{75}{5 \times 5 \times 5}.$$

This root of the denominator is 5; while that of 75 lies between 4 and 5. Adopting 4, we have $\frac{4}{5}$ for the cube root of $\frac{3}{5}$ to within one-fifth. If a greater degree of accuracy be required, we must take the approximate root of 75, by the method we shall soon proceed to explain.

If the denominator be already a perfect square. it will only be necessary to multiply the two terms of the fraction by the square root of this denominator. Thus in order to find the cube root of $\frac{4}{9}$. we multiply the two terms by 3, the square root of 9; we thus obtain

$$\frac{12}{3 \times 3 \times 3}.$$

Taking the root of the greatest cube 8, contained in 12, we have $\frac{2}{3}$ for the root sought, within one third.

CLVIII. It follows from what has been demonstrated in art. 97, that the cube root of a number, which is not a perfect cube, cannot be expressed exactly by any fraction however great may be the denominator; it is therefore an irrational quantity, though not of the same kind with the square root; for it is very seldom that one of them can be expressed by means of the other.

CLIX. We may obtain the approximate cube root by means of vulgar fractions. The mode of proceeding is analogous to that given for finding the square root (103); but, as it may be readily conceived, and is besides not the most eligible, we shall not stop to explain it.

A better method of employing vulgar fractions for this purpose, consists in extracting the root in fractions of a given kind. Thus, if it were required to find, for example, the cube root of 22, within a fifth part of unity, observing that the cube of $\frac{1}{5}$ is $\frac{1}{125}$, we reduce 22 to $\frac{2750}{125}$; then taking the root of 2750, so far as it can be expressed in whole numbers, we have $\frac{14}{5}$, or $2\frac{4}{5}$, for the approximate root of 22.

CLX. It is the practice of most persons, however, in extracting the cube root of a number, by approximation, to convert this number into a decimal fraction, but it is to be observed, that this fraction must be either thousandths or millionths, or of some higher denomination; because when raised to the third power, tenths become thousandths, and thousandths millionths, and in general, *the number of decimal figures found in the cube, is triple the number contained in the root.* From this it is evident, that we must place after the proposed number three times as many ciphers, as there are decimal places required in the root. The root is then to be extracted according to the rules already given, and the requisite number of decimal figures to be distinguished in the result.

If we would find, for example, the cube root of 327, within a hundreth part of unity, we must write six ciphers after this number, and extract the root of 327000000 according to the usual method. This is done in the following manner:

327.000,000	688
216	108
111 000	13872
314 432	
12 568 000	
325 660 672	
1 339 328	

Separating two figures on the right of the result for decimals we have 6,88; but 6,89 would be more exact, because the cube of this last number, although greater than 327, approaches it more nearly than that of 6,88.

If the proposed number contain decimals already, before we proceed to extract the root, we must place on the right as many ciphers, as will be necessary to render the number of decimal figures a multiple of 3. Let there be, for example, 0,07, we must write 0,070, or 70 thousands, which gives for a root, 0,4. In order to arrive at a root exact to hundredths, we must annex three additional ciphers, which gives 0,070000. The root of

the greatest cube contained in 70000 being 41, that of 0,07 becomes 0,41, to within a hundredth.

CLXI. Hitherto we have employed the formula for binomial quantities only in the extraction of the square and cube roots of numbers; this formula leads to an analogous process for obtaining the root of any degree whatever. We shall proceed to explain this process, after offering some remarks upon the extraction of roots, the exponent of which is a divisible number.

We may find the fourth root by extracting the square root twice successively; for by taking first the square root of a fourth power, a^4, for example, we obtain the square, or a^2, the square root of which is a, or the quantity sought.

It is obvious also, that the eighth root may be obtained by extracting the square root three times successively, since the square root of a^8 is a^4, and that of a^4 is a^2, and lastly, that of a^2 is a.

In the same manner it may be shewn, that all roots of a degree, designated by any of the numbers 2, 4, 8, 16, 32, &c., that is, by any power of 2, are obtained by successively extracting the square root.

Roots, the exponents of which are not prime numbers, may be reduced to others of a degree less elevated; the sixth root, for example, may be found by extracting the square and afterwards the cube root. Thus if we take a^6 and go through this process with it, we find by the first step a^3, and by the second a; we may also take first the cube root, which gives a^2, and afterwards the square root, whence we have a, as before.

CLXII. We now proceed to treat of the general method, which we shall apply to roots of the fifth degree. The illustration will be rendered more easy, if we take a particular example; and by comparing the different steps with the methods given for the extraction of the square and the cube root, we shall readily perceive, in what manner we are to proceed in finding roots of any degree whatever.

Let it be required then to extract the fifth root of 231554007. Now the least number, it may be observed, consisting of 2 figures, that is 10, has in its fifth power, which is 100000, six figures; we therefore conclude, that the fifth root of the number proposed contains at least two figures; this root may then be represented by $a + b$, a denoting the tens and b the units. The expression for the proposed number will then be

$$(a + b)^5 = a^5 + 5\,a^4\,b + 10\,a^3\,b^2 + \text{\&c.}$$

We have not developed all the terms of this power, because it is sufficient, as will be seen immediately, that the composition of the first two be known.

Now it is evident, that as a^5, or the fifth power of the tens of this root, can have no figure, that falls below hundreds of thousands, it does not enter into the last five figures on the right of the proposed number; we, therefore, separate these five figures. If there remained more than five figures on the left, we should repeat the same reasoning, and thus separate the proposed number into portions of five figures each, proceeding from the right to the left. The last of these portions on the left, will contain the fifth power of the units of the highest order found in the root.

2315,5 4007	47
1024	
1291 5,4007	1280

We find, by forming the fifth powers of numbers consisting of only one figure, that 2315 lies between the fifth power of 4, or 1024. and that of five, or 1325. We take, therefore, 4 for the tens of the root sought; then subtracting the fifth power of this number, or 1024, from the first portion of the proposed number, we have for a remainder 1291. The remainder, together with the following portion, which is to be brought down, must contain $5\,a^4\,b + 10\,a^3\,b^2 +$ &c. which is left, after a^5 has been subtracted from $(a+b)^5$; but among these terms, that of the highest degree is $5\,a^4\,b$, or five times the fourth power of the tens multiplied by the units, because it has no figure, which falls below tens of thousands. In order to consider this term by itself, we separate the last four figures on the right, which make no part of it, and the number 12915, remaining on the left, will contain this term, together with the tens of thousands arising from the succeeding terms. It is obvious, therefore, that by dividing 12915 by $5\,a^4$, or five times the fourth power of the four tens already found, we shall only approximate the units. The fourth power of 4 is 256; five times this gives 1280; if we divide 12915 by 1280, we find 10 for the quotient, but we cannot put more than 9 in the place of the root, and it is even necessary, before we adopt this, to try whether the whole root 49, which we thus obtain, will not give a fifth power greater than the proposed number. We find indeed, by pursuing this course, that the number 49 must be diminished by two units, and that the actual root is 47, with a remainder 2209000; for the fifth power of 47 is 229345007; that is, the exact root of the proposed number falls between 47 and 48.

If there were another portion still, we should bring it down and annex it to the remainder, resulting from the subtraction of the fifth power found as above, from the first two portions, and

proceed with this whole remainder, as we did with the preceding, and so on.

After what has been said, it will be easy to apply the rules, which have been given, as well in extracting the square and cube root of fractions, as in approximating the roots of imperfect powers of these degrees.

CLXIII. We may by processes, founded on the same principles, extract the roots of literal quantities. The following example will be sufficient to illustrate the method, which is to be employed, whatever be the degree of the root required.

We found in art. 151, the sixth power of $2x^3 - 5a^3$; we shall now extract the root of this power. The process is as follows:

$$\begin{array}{l|l}
64x^{18} - 960a^3x^{15} + 6000a^6x^{12} - 20000a^9x^9 & 2x^3 - 5a^3 \\
\qquad\qquad + 37500a^{12}x^6 - 37500a^{15}x^3 & \overline{192x^{15}} \\
\qquad\qquad\qquad + 15625a^{18} & \\
-64x^{18} & \\
\hline
\text{rem.}\quad -960a^3x^{15} + \&c. &
\end{array}$$

The quantity proposed being arranged with reference to the letter x, its first term must be the sixth power of the first term of the root arranged with reference to the same letter; taking then the sixth root of $64x^{18}$, according to the rule given in art. 145, we have $2x^3$ for the first term of the root required.

If we raise this result to the sixth power, and subtract it from the proposed quantity, the remainder must necessarily commence with the second term, produced by the development of the sixth power of the first to terms of the root. But, in the expression

$$(a + b)^6 = a^6 + 6a^5b + \&c.$$

this second term is the product of six times the fifth power of the first term of the root by the second; and if we divide it by $6a^5$, the quotient will be the second term b.

We must, therefore, take six times the fifth power of the

$$6 \times 32x^{15} \text{ or } 192x^{15},$$

and divide, by this quantity, the term $-960a^3x^{15}$, which is the first term of the remainder, after the preceding operation; the quotient $-5a^3$ is the second term of the root. In order to verify it, we raise the binomial $2x^3 - 5a^3$ to the sixth power, which we find is the proposed quantity itself.

If the quantity were such as to require another term in the root, we should proceed to find, after the manner above given, a second remainder, which would begin with six times the

product of the fifth power of the first two terms of the root by the third, and which consequently being divided by $6\,(2\,x^3 - 5\,a^3)^5$, the quotient would be this third term of the root; we should then verify it by taking the sixth power of the three terms. The same course might be pursued, whatever number of terms might remain to be found.

Of Equations with Two Terms.

CLXIV. Every equation, involving only one power of the unknown quantity combined with known quantities, may always be reduced to two terms, one of which is made up of all those which contain the unknown quantity, united in one expression, and the other comprehends all the known quantities collected together. This has been already shewn with respect to equations of the second degree, art. 105, and may be easily proved concerning those of any degree whatever.

If we have, for example, the equation

$$a^2\,x^5 - a^5\,b^2 = b^4\,c^3 + a\,c\,x^5,$$

by bringing all the terms involving x into one member, we obtain

$$a^2\,x^5 - a\,c\,x^5 = b^4\,c^3 + a^5\,b^2,$$

or

$$(a^2 - a\,c)\,x^5 = b^4\,c^3 + a^5\,b^2$$

Now if we represent the quantities

$$a^2 - a\,c \text{ by } p,\ b^4\,c^3 + a^5\,b^2 \text{ by } q,$$

the preceding equation becomes

$$p\,x^5 = q\,;$$

freeing x^5 from the quantity, by which it is multiplied, we have

$$x^5 = \frac{q}{p}\,;$$

whence we conclude

$$x = \sqrt[5]{\frac{q}{p}}.$$

In general, every equation with two terms being reduced to the form

$$p\,x^m = q,$$

gives

$$x^m = \frac{q}{p}\,;$$

taking the root then of the degree m of each member, we have

$$x = \sqrt[m]{\frac{q}{p}}.$$

CLXV. It must be observed, that if the exponent m is an odd number, the radical expression will have only one sign, which will be that of the original quantity (139).

When the exponent m is even, the radical expression will have the double sign $\pm$; it will in this case be imaginary, if the quantity $\frac{q}{p}$ is negative, and the question will be absurd, like those of which we have seen examples in equations of the second degree (139).

See some examples.

The equation $x^5 = -1024$,

gives

$$x = \sqrt[5]{-1024} = -4,$$

the exponent 5 being an odd number.

The equation

$$x^4 = 625,$$

gives $x = \pm \sqrt[4]{625} = \pm 5$,

as the exponent 4 is even.

Lastly, the equation

$$x^4 = -16,$$

which gives

$$x = \pm \sqrt[4]{-16},$$

leads only to imaginary values, because while the exponent 4 is even, the quantity under the radical sign is negative.

CLXVI. We shall here notice an analytical fact, which deserves attention on account of its great utility, in the remaining part of the present treatise, and which is sufficiently remarkable in itself; it is this, that all the expressions $x - a$, $x^2 - a^2$, $x^3 - a^3$ and in general $x^m - a^m$ (m being any positive whole number), are exactly divisible by $x - a$. This is obvious with respect to the first. We know that the second

$$x^2 - a^2 = (x + a)(x - a) \text{ (34)},$$

and the others may be easily decomposed by division. If we divide $x^m - a^m$ by $x - a$, we obtain for a quotient

$$x^{m-1} + ax^{m-2} + a^2x^{m-3} + \&c.$$

the exponent of x, in each term, being less by unity than in the preceding, and that of a increasing in the same ratio. But instead of pursuing the operation through its several steps, we shall present immediately to view the equation

$$\frac{x^m - a^m}{x - a} = x^{m-1} + ax^{m-2} + a^2x^{m-3} \ldots\ldots + a^{m-1}x + a^{m-1},$$

which may be verified by multiplying the second member by $x - a$. It then becomes

$$\begin{array}{l} x^m + a\,x^{m-1} + a^2x^{m-2} \ldots\ldots\ldots\ldots\ldots + a^{m-2}x^2 + a^{m-1}x \\ \quad - a\,x^{m-1} - a^2x^{m-2} - a^3x^{m-3} \ldots\ldots\ldots\ldots - a^{m-1}x - a^m; \end{array}$$

all the terms in the upper line, after the first, being the same, with the exception of the signs, as those preceding the last in the lower line, there only remains after reduction $x^m - a^m$, that is, the dividend proposed.

It must be observed, that the term a^2x^{m-2}, in the upper line, is necessarily followed by the term a^3x^{m-3}, which is destroyed by the corresponding term in the lower line; and that, in the same manner we find, in the lower line, before the term $a^{m-1}x$, a term $-a^{m-2}x^2$, which destroys the corresponding one in the upper line. These terms are not expressed, but are supposed to be comprehended in the interval denoted by the points.

CLXVII. This leads to very important consequences, relative to the equation with two terms $x^m = \frac{q}{p}$.

If we designate by a the number, which is obtained by directly extracting the root according to the rules given in art. 162, we have

$$\frac{q}{p} = a^m \quad \text{or} \quad x^m = a^m;$$

transposing the second member, we obtain

$$x^m - a^m = 0.$$

The quantity $x^m - a^m$ is divisible by $x - a$, and we have by the preceding article

$$x^m - a^m = (x - a)\,(x^{m-1} + a\,x^{m-2} \ldots\ldots + a^{m-2}x + a^{m-1}).$$

This last result, which vanishes when $x = a$, is also reduced to nothing, if we have

$$x^{m-1} + a\,x^{m-2} \ldots\ldots + a^{m-2}x + a^{m-1} = 0. \quad (120);$$

and, consequently, if there exists a value of x, which satisfies this last equation, it will satisfy also the equation proposed.

These values have with unity very simple relations, which may be discovered by making $x = a y$; then the equation $x^m - a^m = 0$ becomes

$$a^m y^m - a^m = 0, \quad \text{or} \quad y^m - 1 = 0,$$

and we obtain the values of x, by multiplying those of y by the number a.

The equation $y^m - 1 = 0$, gives in the first place

$$y^m = 1, \; y = \sqrt[m]{1} = 1;$$

then by dividing $y^m - 1$ by $y - 1$, we have

$$y^{m-1} + y^{m-2} + y^{m-3} \ldots\ldots + y^2 + y + 1.$$

Taking this quotient for one of the members, and zero for the other, we form the equation on which the other values of y depend; and these values will, in the same manner, satisfy the equation

$$y^m - 1 = 0, \quad \text{or} \quad y^m = 1,$$

that is, their power of the degree m will be unity.

Hence we infer the fact, singular at first view, that unity may have many *roots* beside itself. These roots, though imaginary, are still of frequent use in analysis. We can, however, exhibit here only those of the four first degrees, as it is only for these degrees, that we can resolve, by preceding observations, the equation

$$y^{m-1} + y^{m-2} \ldots\ldots + 1 = 0,$$

from which they are derived.

(1.) Let $m = 2$, we have

$$y^2 - 1 = 0,$$

whence we obtain

$$y = +1, \quad y = -1.$$

(2.) By making $m = 3$, we have

$$y^3 - 1 = 0,$$

whence we deduce

$$y = 1.$$

then $$y^2 + y + 1 = 0.$$

This last equation being resolved, gives

$$y = \frac{-1 + \sqrt{-3}}{2}, \quad y = \frac{-1 - \sqrt{-3}}{2};$$

thus we have for this degree the three roots

$$y = 1, \quad y = \frac{-1 + \sqrt{-3}}{2}, \quad y = \frac{-1 - \sqrt{-3}}{2}.$$

The last two are imaginary; but if we take the cube, forming that of the numerator, by the rule given in art. 84, and observing that the square of $\sqrt{-3}$ being -3, its cube is $-3\sqrt{-3}$, we still find $y^3=1$, in the same manner as when we employ the root $y=1$.

(3.) Taking $m=4$, we have

$$y^4-1=0,$$

from which we deduce

$$y=1,$$

then

$$y^3+y^2+y+1=0,$$

which can be expressed by

$$y^2(y+1)+y+1=(y+1)(y^2+1)=0,$$

from which we get

$$y+1=0 \text{ or } y^2+1=0;$$

these equations give

$$y=-1,\ y=+\sqrt{-1},\ y=-\sqrt{-1}$$

the four roots then of the proposed equation are

$$y=+1,\ y=-1,\ y=+\sqrt{-1},\ y=-\sqrt{-1}$$

of these four values, only the two first are real, the other two are imaginary.

These values are also found by observing that

$$y^4-1=(y^2+1)(y^2-1),$$

whence we have successively

$$y^2-1=0,\quad y^2+1=0,$$

consequently

$$y=+1,\quad y=-1,\quad y=+\sqrt{-1},\quad y=-\sqrt{-1}.$$

This multiplicity of roots of unity is agreeable to a general law of equations, according to which any unknown quantity admits of as many values, as there are units in the exponent denoting the degree of the equation, by which this unknown quantity is determined; and when the question does not admit of so many real solutions, the number is completed by purely algebraic symbols, which being subjected to the operations, that are indicated, verify the equation.

Hence it follows, that there are two kinds of expressions or values for the roots of numbers; the first, which we shall term the *arithmetical determination*, is the number which is found by the methods explained in art. 162, and which answers to each particular case; the second comprehends negative values

and imaginary expressions, which we shall designate by the term *algebraic determinations*, because they consist merely in the combination of algebraic signs.

Of Equations which may be resolved in the same manner as those of the Second Degree.

CLXVIII. These are equations, which contain only two different powers of the unknown quantity, the exponent of one of which is double that of the other. Their general formula is

$$x^{2m} + p\,x^m + q = 0,$$

p and q being known quantities.

Now if we take x^m for the unknown quantity, and make $x^m = u$, we have

$$x^{2m} = u^2,$$

whence

$$u^2 + p\,u = q,$$

$$u = \frac{-p \pm \sqrt{p^2 - 4q}}{2}, \quad (113);$$

restoring x^m in the place of u, we have

$$x^m = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

an equation consisting of two terms, since the expression

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

as it implies only known operations, to be performed on given quantities, must be regarded as representing known quantities.

After a little practice, these substitutions are not made use of, as we can draw directly from the equation

$$x^{2m} + p\,x^m + q = 0$$

$$x^m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}, \therefore x = \pm \sqrt[m]{\frac{-p \pm \sqrt{p^2 - 4q}}{2}}$$

even the intermediate step,

$$x^m = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

is more commonly omitted.

Designating the two values of this expression by a and a', we have

$$x^m = a \text{ and } x^m = a',$$

from which we obtain

$$x = \sqrt[m]{a} \text{ and } x = \sqrt[m]{a'}.$$

If the exponent m be even, instead of the two values given above, we shall have four, since each radical expression may take the sign $\pm$; then

$$x = +\sqrt[m]{a},\ x = +\sqrt[m]{a'},$$
$$x = -\sqrt[m]{a},\ x = -\sqrt[m]{a'},$$

and these four values will be real, if the quantities a and a' are positive.

All the values of x may be comprehended under one formula, by indicating directly the root of the two members of the equation

$$x^m = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

which gives

$$x = \pm \sqrt[m]{\frac{-p \pm \sqrt{(p^2 - 4q.)}}{2}}$$

The following question produces an equation of this kind.

CLXIX. *To resolve the number 6 into two such factors, that the sum of their cubes shall be 35.*

Let x be one of these factors, the other will be $\frac{6}{x}$; then taking the sum of their cubes x^3 and $\frac{216}{x^3}$, we have the equation

$$x^3 + \frac{216}{x^3} = 35,$$

which may be reduced to

$$x^6 + 216 = 35\,x^3$$

or

$$x^6 - 35\,x^3 = -216.$$

If we consider x^3 as the unknown quantity, we obtain, by the rule given for equations of the second degree,

$$x^3 = \tfrac{35}{2} \pm \sqrt{(\tfrac{35}{2})^2 - 216}.$$

By going through the numerical calculations, which are indicated, we find

$$(\tfrac{35}{2})^2 = \tfrac{1225}{4},$$
$$\sqrt{(\tfrac{35}{2})^2 - 216} = \sqrt{\tfrac{361}{4}} = \tfrac{19}{2},$$

and consequently,

$$x^3 = \begin{cases} \tfrac{35}{2} + \tfrac{19}{2} = \tfrac{54}{2} = 27, \\ \tfrac{35}{2} - \tfrac{19}{2} = \tfrac{16}{2} = 8, \end{cases}$$

whence, $\sqrt[3]{27} =$

$\sqrt[3]{8} = 2.$

The first value gives for the second factor $\frac{6}{3}$ or 2, while the second value presents $\frac{6}{2}$ or 3; we have, therefore, in the one case 3 and 2 for the factors sought, and in the other 2 and 3. These two solutions differ only in the order of the factors of the given number 6.

CLXX. The equations, we have been considering, are also comprehended under the general law given in art. 167; for the values of $\sqrt[m]{a}$, $\sqrt[m]{a'}$ are to be multiplied by the roots of unity belonging to the degree denoted by the exponent m.

Applying what has been said to the equation,

$$x^6 - 35\,x^3 = -216,$$

we find the six following roots;

$$x = 1 \times 3, \qquad x = 1 \times 2,$$

$$x = \frac{-1 + \sqrt{-3}}{2} \times 3, \quad x = \frac{-1 + \sqrt{-3}}{2} \times 2,$$

$$x = \frac{-1 - \sqrt{-3}}{2} \times 3, \quad x = \frac{-1 - \sqrt{-3}}{2} \times 2,$$

of which the first two only are real.

Calculus of Radical Expressions.

CLXXI. The great number of cases, in which no exact root can be found, and the length of the operation necessary for obtaining it by approximation, have led algebraists to endeavour to perform immediately upon the quantities subjected to the radical sign, the fundamental operations intended to be performed upon their roots. In this way we simplify the expression as much as possible, and leave the extracting of the root, which is a more complicated process, to be performed last, when the quantities are reduced to the most simple state, which the nature of the question will allow.

The addition and subtraction of dissimilar radical quantities can take place only by means of the signs + and —. For example, the sums

$$\sqrt[3]{a} + \sqrt[5]{a}, \quad \sqrt[3]{a} + \sqrt[3]{b},$$

and the differences

$$\sqrt[3]{a} - \sqrt[5]{a}, \quad \sqrt[3]{a} - \sqrt[3]{b},$$

can be expressed only under their present form.

The same cannot be said of the expression

$$4\,a\sqrt[3]{2\,b}+\sqrt[3]{16\,a^3b}-\frac{5c}{ad}\sqrt[3]{2\,a^6b},$$

because the radical quantities, of which it is composed, become similar, when they are reduced to their more simple forms, according to the method explained in art. 130. First, we have

$$\sqrt[3]{16\,a^3b}=\sqrt[3]{8\,a^3,\,2b}\;=2\,a\sqrt[3]{2\,b}$$
$$\sqrt[3]{2\,a^6\,b}=\sqrt[3]{a^6.\,2b}\;=\;a^2\sqrt[3]{2\,b};$$

the quantity, therefore, becomes

$$4\,a\sqrt[3]{2\,b}+2\,a\sqrt[3]{2\,b}-\frac{5\,a^2c}{ad}\sqrt[3]{2\,b},$$

which gives, when reduced,

$$6\,a\sqrt[3]{2\,b}-\frac{5ac}{d}\sqrt[3]{2\,b},\ \text{or}\ \ (6\,d-5\,c)\frac{a}{d}\sqrt[3]{2\,b}.$$

CLXXII. With respect to other operations, the calculus of radical quantities depends upon the principle already referred to, namely; *that a product, consisting of several factors, is raised to any power by raising each of the factors to this power*. So also, by suppressing the radical sign, prefixed to a quantity, we raise this quantity to the power denoted by the exponent of this sign. For example, $\sqrt[7]{a}$ raised to the seventh power, is a simply, since this operation, being the reverse of that which is indicated by the sign $\sqrt[5]{\ }$, merely restores the quantity a to its original state.

According to the principles here laid down, if, for example, in the expression

$$\sqrt[7]{a}\times\sqrt[7]{b},$$

we suppress the radical signs, the result $a\,b$ will be the seventh power of the above product; and taking the seventh root, we find

$$\sqrt[7]{a}\times\sqrt[7]{b}=\sqrt[7]{a\,b}.$$

This reasoning, which may be applied to all similar cases, shews, that in order *to multiply two radical expressions of the same degree together, we must take the product of the quantities under the radical sign, observing to place it under a sign of the same degree.*

We have by this rule

$$3\sqrt{2\,a\,b^3}\times7\sqrt{5\,a^3\,b\,c}=21\sqrt{10\,a^4\,b^4\,c}=$$
$$21\,a^2\,b^2\sqrt{10\,c};$$

$$\sqrt[4]{a^2-b^2}\times\sqrt[4]{a^2+b^2}=\sqrt[4]{(a^2-b^2)(a^2+b^2)}=$$
$$\sqrt[4]{a^4-b^4};$$

$$\sqrt[5]{\frac{2a^9-a^3b^6}{a^4-b^4}}\times\sqrt[5]{\frac{a^2b^3c^2+b^5c^2}{d^2}}$$

$$=\sqrt[5]{\frac{2a^9-a^3b^6}{a^4-b^4}\times\frac{a^2b^3c^2+b^5c^2}{d^2}}$$

$$=\sqrt[5]{\frac{a^3(2a^6-b^6)}{a^4-b^4}\times\frac{b^3c^2}{d^2}(a^2+b^2)}$$

$$=\sqrt[5]{\frac{a^3b^3c^2}{d^2}\times\frac{2a^6-b^6}{a^2-b^2}},$$

since

$$a^4-b^4=(a^2+b^2)(a^2-b^2).$$

CLXXIII. As the seventh power of the expression $\frac{\sqrt[7]{a}}{\sqrt[7]{b}}$, for example, is $\frac{a}{b}$, it will be seen, by taking the seventh root of this last result, that

$$\frac{\sqrt[7]{a}}{\sqrt[7]{b}}=\sqrt[7]{\frac{a}{b}}.$$

Hence *to divide a radical quantity by another of the same degree, we must take the quotient arising from the division of the quantities under the radical sign, recollecting to place it under a sign of the same degree.*

We find by this rule, that

$$\frac{\sqrt{6ab}}{\sqrt{3a}}=\sqrt{\frac{6ab}{3a}}=\sqrt{2b};$$

$$\frac{\sqrt{a^2-b^2}}{\sqrt{a+b}}=\sqrt{\frac{a^2-b^2}{a+b}}=\sqrt{a-b};$$

$$\frac{\sqrt[5]{a^4b}}{\sqrt[5]{b^3c^2}}=\sqrt[5]{\frac{a^4b}{b^3c^2}}=\sqrt[5]{\frac{a^4}{b^2c^2}}.$$

CLXXIV. It follows from the rule, given in art. 172, for the multiplication of radical quantities of the same degree, *that to raise a radical quantity to any power whatever, we have only to raise to this power the quantity under the radical sign, observing that the result must take the same sign:* thus to raise $\sqrt[5]{a\,b}$, for example to the third power is to take the product

$$\sqrt[5]{a\,b} \times \sqrt[5]{a\,b} \times \sqrt[5]{a\,b},$$

and as the radical signs are all of the same degree, the quantities to which they belong are to be multiplied together, and the radical sign to be prefixed to the product, which gives

$$\sqrt[5]{5^{3}\,b^{3}}.$$

In the same manner $\sqrt[7]{a^2\,b^3}$ raised to the fourth power, gives $\sqrt[7]{a^8\,b^{12}}$, which may be reduced to

$$a\,b\,\sqrt[7]{a\,b^5},$$

by resolving $a^8\,b^{12}$ into $a^7\,b^7 \times a\,b^5$, and taking the root of the factor $a^7\,b^7$ (138).

It may be observed, that *when the exponent belonging to the radical sign is divisible by that of the power to which the proposed quantity is to be raised, the operation is performed by dividing the first exponent by the second.* For example,

$$(\sqrt[6]{a})^2 = \sqrt[3]{a},$$

because $\frac{6}{2} = 3$.

Indeed $\sqrt[6]{a}$ denotes a quantity, which is six times a factor in a, and the quantity $\sqrt[3]{a}$, which is obtained by dividing 6 by 2, being only three times a factor in a, is consequently equivalent to the product of two of the first factors, and is therefore the second power of one of these factors, or of $\sqrt[6]{a}$.

The same reasoning may be applied to all similar cases, as in the following example;

$$\sqrt[12]{(a^2 b)^3} = \sqrt[4]{a^2\,b}.$$

CLXXV. If we reverse the methods given in the preceding article, we shall be furnished with rules for extracting the roots of radical quantities.

We perceive by attending to the rule first stated, that *if the exponents of the quantities under the radical sign are divisible by that of the root required, the operation may be performed as if there were no radical sign, only it is to be observed, that the result must be placed under the original sign.*

We find, for example, that

$$\sqrt[3]{\sqrt[5]{a^6}} = \sqrt[5]{\sqrt[3]{a^6}} = \sqrt[5]{a^2}.$$

$$\sqrt[4]{\sqrt[3]{a^4b^8}} = \sqrt[3]{\sqrt[4]{a^4b^8}} = \sqrt[3]{ab^2}.$$

From the second rule given in the preceding article, it is evident, that *the general method for finding the root of radical quantities, is to multiply the exponent belonging to the radical sign by that of the root, which is to be extracted.*

By this last rule, we find, that

$$\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4}.$$

Indeed, $\sqrt[5]{a^4}$ is a quantity, which is five times a factor in a^4 (24, 137); but the cube root of $\sqrt[5]{a^4}$, being also three times a factor in this last quantity, is found 5×3 times or 15 times a factor in the first a^4; therefore $\sqrt[3]{\sqrt[5]{a^4}} = \sqrt[15]{a^4}$. In the same manner it might be shewn, that $\sqrt[5]{\sqrt[3]{a^4}} = \sqrt[15]{a^4}$.

CLXXVI. Since by multiplying the exponent of a quantity under a radical sign, by any number (174), we raise the root which is indicated, to the power denoted by this number, and by multiplying also the exponent belonging to the radical sign by the same number (175), we obtain for the result a root of a degree equal to that of the power which was before formed, it is evident, that this second operation reduces the proposed quantity back to its original state.

The expression $\sqrt[5]{a^3}$, for example, may be changed into $\sqrt[35]{a^{21}}$, by multiplying the exponents 5 and 3 by 7; for multiplying the exponent of a^3 by 7, we have, making use of the radical sign, $\sqrt[5]{a^{21}}$, the seventh power of the proposed radical quantity, and multiplying by 7 the exponent 5 belonging to the radical sign in the expression $\sqrt[5]{a^{21}}$, we obtain the seventh root of the former result; this last process, therefore, restores the expression to its original value.

CLXXVII. By this double operation, *we reduce to the same degree any number of radical quantities of different degrees, by multiplying, at the same time, the exponent belonging to each radical sign, and those of the quantities under this sign, by the product of the exponents belonging to all the other radical signs.* That the new exponents, which are thus

found for the radical signs, are the same, is obvious at once, since they arise from the product of all the exponents belonging to the original radical signs; and after what has been said above, it is evident that the value of each radical quantity is the same as before.

By this rule we transform

$$\sqrt[5]{a^3 b^2} \text{ and } \sqrt[7]{c^4 d^3},$$

into

$$\sqrt[35]{a^{21} b^{14}} \text{ and } \sqrt[35]{c^{20} d^{15}}.$$

In the same manner, the three quantities,

$$\sqrt[3]{a b^2},\ \sqrt[5]{a^4 c^3},\ \sqrt[7]{b^4 c^8},$$

become respectively

$$\sqrt[105]{a^{35} b^{70}},\ \sqrt[105]{a^{42} c^{63}},\ \sqrt[105]{b^{60} c^{45}}.$$

If we meet with numbers, under the radical signs, we shall be led, in applying this rule, to raise them to the power denoted by the product of the exponents belonging to the other radical signs.

CLXXVIII. In the same way, we may *place under a radical sign a factor which is without one, by raising it to the power denoted by the exponent which accompanies this sign.*

We may change, for example,

$$a^2 \text{ into } \sqrt[5]{a^{10}}, \text{ and } 2a\sqrt[3]{b} \text{ into } \sqrt[3]{8 a^3 b}.$$

CLXXIX. After having, by the transformation explained above, reduced any radical quantities whatever, to the same degree, we may apply to them the rules, given in articles 172, and 173, for the multiplication and division of radical quantities of the same degree.

Let there be the general expressions

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s};$$

we change (169)

$$\sqrt[m]{a^p b^q},\ \sqrt[n]{b^r c^s},$$

into

$$\sqrt[mn]{a^{np} b^{nq}},\ \sqrt[mn]{b^{mr} c^{ms}},$$

then by the rule given in art. 172, we have

$$\sqrt[mn]{a^{np} b^{nq}} \times \sqrt[mn]{b^{mr} c^{ms}} = \sqrt[mn]{a^{np} b^{nq+mr} c^{m}},$$

for the product of the proposed radical quantities.

We have also by the rule, art. 173.

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{\sqrt[mn]{a^{np} b^{nq}}}{\sqrt[mn]{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq}}{b^{mr} c^{ms}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

Remarks on some peculiar cases, which occur in the Calculus of Radical Quantities.

CLXXX. The rules of which we have reduced the calculus of radical quantities, may be applied without difficulty, when the quantities employed are real. But they might lead the learner into error with regard to imaginary quantities, if they are not accompanied with some remarks upon the properties of equations with two terms.

For example, the rule laid down in art. 172, gives directly

$$\sqrt{-a} \times \sqrt{-a} = \sqrt{-a \times -a} = \sqrt{a^2};$$

and if we take $+a$ for $\sqrt{a^2}$, we evidently come to an erroneous result, for the product $\sqrt{-a} \times \sqrt{-a}$, being the square of $\sqrt{-a}$, must be obtained by suppressing the radical sign, and is therefore equal to $-a$.

This difficulty is obviated by observing, that when we do not know by what method the square a^2 has been formed, we must assign for its root both $+a$ and $-a$; but when, by means of steps already taken, we know which of these two quantities multiplied by itself produced a^2, we are not allowed, in going back to the root, to take the other quantity. This is evidently the case with respect to the expression $\sqrt{-a} \times \sqrt{-a}$; here we know, that the quantity a^2, contained under the radical sign in the expression $\sqrt{a^2}$, arises from $-a$ multiplied by $-a$; the ambiguity, therefore, is prevented, and it will be readily seen, that in taking the root, we are limited to $-a$.

The difficulty above mentioned would present itself in regard to the product $\sqrt{a} \times \sqrt{a}$, if we were not led by the circumstance of there being no negative sign in the expression, to take immediately the positive value of $\sqrt{a^2}$. In this case, since a^2 arises from $+a$ multiplied by $+a$, its root must necessarily be $+a$.

There can be no doubt with respect to examples of the kind we have been considering; but there are cases, which can be clearly explained only by attending to the properties of equations with two terms.

CLXXXI. If, for example, it were required to find the product $\sqrt[4]{a}\ \sqrt{-1}$; reducing the second of these radical expressions to the same degree with the first (177), we have

$$\sqrt[4]{a} \times \sqrt[4]{(-1)^2} \quad \sqrt[4]{a} \times \sqrt[4]{+1} = \sqrt[4]{a},$$

a result which is real, although it appears evident, that the quantity $\sqrt[4]{a}$ multiplied by the imaginary quantity $\sqrt{-1}$ ought to give an imaginary product. It must not be supposed, however, that the expression $\sqrt[4]{a}$ is in all respects false, but only that it is to be taken in a very peculiar sense.

In fact, $\sqrt[4]{a}$, considered algebraically, being the expression for the unknown quantity x, in the equation with two terms,

$$x^4 - a = 0,$$

admits of four different values (167); for if we make $a = \alpha^4$, by taking α to represent the numerical value of $\sqrt[4]{a}$, considered independently of its sign, or the arithmetical determination of this quantity, we have the four values

$$\alpha \times +1,\ \alpha \times -1,\ \alpha \times +\sqrt{-1},\ \alpha \times -\sqrt{-1},$$

the third of which is precisely the product proposed.

By a little attention, it will be readily perceived, whence the ambiguity, of which we have been speaking arises. The second power, $+1$ of the quantity -1 under the radical sign, as it may arise as well from $+1 \times +1$, as from -1×-1, causes the quantity $\sqrt[4]{1}$ to have two values, which are not found in $\sqrt{-1}$.

In general, the process by which the product $\sqrt[m]{a} \times \sqrt[n]{b}$ is formed, is reduced to that of raising this product to the power $m\,n$; for if we represent it by z, that is, if we make

$$\sqrt[m]{a} \times \sqrt[n]{b} = z,$$

by raising the two members of this equation, first to the power m, we have

$$a \sqrt[n]{b^m} = z^m,$$

again, raising it to the power n, we obtain

$$a^n b^m = z^{mn} \text{ or } z = \sqrt[mn]{a^n b^m}.$$

This product, therefore, being determined only by means of its power of the degree $m\,n$, or by an equation of this degree with two terms, must have $m\,n$ values (167). This will be perceived at once, if we reflect that the expressions $\sqrt[m]{a}$ and $\sqrt[n]{b}$, being nothing but the values of the unknown quantities x and y in the equations with two terms,

2 O 2

$$x^m - a = 0, \; y^n - b = 0,$$

$$\therefore (x\,y)^{mn} = x^{mn}\,y^{mn} = a^n\,b^m \therefore x\,y = \sqrt[mn]{a^n\,b^m};$$

and, consequently, admitting of m and of n determinations, we have, by uniting the several m determinations of x, with the several n determinations of y, $m\,n$ determinations of the product required.

When we are employed upon real quantities, there is no difficulty in finding the values, because the number of those, that are real, is never more than two (165), which differ only in the sign.

CLXXXII. If we use the transformation explained in art. 167, the difficulty will be confined to the roots of $+1$ and -1; for if we make $x = \alpha\,t$ and $y = \beta\,u$, α and β denoting the numerical values of $\sqrt[m]{a}$, $\sqrt[n]{b}$ considered without regard to the sign, the equations

$$x^m \mp a = 0, \qquad y^n \mp b = 0,$$

become

$$t^m \mp 1 = 0, \qquad u^n \mp 1 = 0,$$

whence

$$x\,y = \sqrt[m]{\pm a} \times \sqrt[n]{\pm b} = \alpha\,\beta\,t\,u = \alpha\,\beta\,\sqrt[m]{\pm 1} \times \sqrt[n]{\pm 1};$$

in which $\alpha\,\beta$ represents the product of the numbers $\sqrt[m]{a}$, $\sqrt[n]{b}$, or the arithmetical determination of the root of the degree $m\,n$ of the number $a^n\,b^m$.

If we would give a determinate value to the product of the radical quantities $\sqrt[m]{\pm a}$, $\sqrt[n]{\pm b}$, by fixing the degree of the radical signs, we must obtain from the equations

$$t^m \mp 1 = 0, \quad u^n \mp 1 = 0,$$

the several expressions for $\sqrt[m]{\pm 1}$, $\sqrt[n]{\pm 1}$, and combine them in a suitable manner*.

To conclude, these operations are not often required, except in some very simple cases, of which the following are the principal;

* When the exponent m, is an odd number, then $\sqrt[m]{-1} = -\sqrt[m]{+1}$; but when an even number, it does not hold. Making, for example, $m = 4$ we find that

$$y^4 + 1 = (y^2 + y\sqrt{2} + 1)\,(y^2 - y\sqrt{2} + 1)$$

equalising each of these factors to zero, we obtain the 4 expressions of $\sqrt[4]{-1}$.

(1.) $\sqrt{-a}\times\sqrt{-b}=\sqrt{a}\times\sqrt{b}\,(\sqrt{-1}\times\sqrt{-1})$;

We suppress the radical sign in the expression $\sqrt{-1}$, and obtain

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{a\,b}\times -1=-\sqrt{a\,b}.$$

(2.) $\sqrt[4]{-a}\times\sqrt[4]{-b}=\sqrt[4]{a\,b}\,(\sqrt[4]{-1})^2$;

We do not here multiply -1 by -1, because this would lead to the ambiguity mentioned in art. 181; but observing, that the square of the fourth root is simply the square root, we have

$$\sqrt[4]{-a}\times\sqrt[4]{-b}=\sqrt[4]{a\,b}\times\sqrt[4]{-1}.$$

(3.) $\sqrt[6]{-a}\times\sqrt[6]{-b}=\sqrt[6]{a\,b}\times(\sqrt[6]{-1})^2$

$$=\sqrt[6]{a\,b}\times\sqrt[3]{-1}=\sqrt[6]{a\,b}\times -1=-\sqrt[6]{a\,b}.$$

The results will be thus found to be alternately real and imaginary.

Calculus of Fractional Exponents.

CLXXXIII. If we substitute in the place of the radical signs, their corresponding fractional exponents (140), and apply immediately the rules for the exponents, we shall obtain the same results, as those furnished by the methods employed in the calculus of radical quantities.

If we transform, for example,

$$\sqrt[5]{a^3\,b^2},\ \sqrt[5]{a^3\,c^2},$$

into

$$a^{\frac{3}{5}}\,b^{\frac{2}{5}},\quad a^{\frac{3}{5}}\,c^{\frac{2}{5}},$$

we have

$$\sqrt[5]{a^3\,b^2}\times\sqrt[5]{a^3\,c^2}=a^{\frac{3}{5}}\,b^{\frac{2}{5}}\times a^{\frac{3}{5}}\,c^{\frac{2}{5}}=$$

$$a^{\frac{3}{5}+\frac{3}{5}}\,b^{\frac{2}{5}}c^{\frac{2}{5}}=a^{\frac{6}{5}}\,b^{\frac{2}{5}}\,c^{\frac{2}{5}};$$

then, since $\frac{6}{5}=1+\frac{1}{5}$, and consequently,

$$a^{\frac{6}{5}}=a^{1+\frac{1}{5}}=a\times a^{\frac{1}{5}}\ (25),$$

and $a^{\frac{1}{5}}\,b^{\frac{2}{5}}\,c^{\frac{2}{5}}$ is equivalent to $\sqrt[5]{a\,b^2\,c^2}$ we have

$$\sqrt[5]{a^3\,b^2}\times\sqrt[5]{a^3\,c^2}=a\sqrt[5]{a\,b^2\,c^2},$$

a result which is not only exact, but is reduced to its most simple form.

Let there be the general example $\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s}$; the radical expressions here employed may be transformed into

$$a^{\frac{p}{m}} b^{\frac{q}{m}},\quad b^{\frac{r}{n}} c^{\frac{s}{n}};$$

we then have, according to the rules for exponents, (25),

$$a^{\frac{p}{m}} b^{\frac{q}{m}} \times b^{\frac{r}{n}} c^{\frac{s}{n}} = a^{\frac{p}{m}} b^{\frac{q}{m}+\frac{r}{n}} c^{\frac{s}{n}}.$$

Now in order to add the fractions $\frac{q}{m}$, $\frac{r}{n}$, we must reduce them to the same denominator; and to give uniformity to the results, we must do the same with respect to the fractions $\frac{p}{m}$, $\frac{s}{n}$; we obtain by this means,

$$a^{\frac{np}{mn}} b^{\frac{nq+mr}{mn}} c^{\frac{ms}{mn}};$$

and placing this result under the radical sign, we have

$$\sqrt[m]{a^p b^q} \times \sqrt[n]{b^r c^s} = \sqrt[nm]{a^{np} b^{nq+mr} c^{ms}}.$$

CLXXXIV. The manner of performing division is equally simple, we have for example

$$\frac{\sqrt[5]{a^3 b^2}}{\sqrt[5]{a^4 c}} = \frac{a^{\frac{3}{5}} b^{\frac{2}{5}}}{a^{\frac{1}{5}} c^{\frac{1}{5}}} = \frac{b^{\frac{2}{5}}}{a^{\frac{4}{5}-\frac{3}{5}} c^{\frac{1}{5}}} \quad (38),$$

which may be reduced to

$$\frac{b^{\frac{2}{5}}}{a^{\frac{1}{5}} c^{\frac{1}{5}}};$$

this placed under the radical sign becomes

$$\frac{\sqrt[5]{a^3 b^4}}{\sqrt[5]{a^4 c}} = \sqrt[5]{\frac{b^4}{ac}}.$$

We have in general

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m}}}{b^{\frac{r}{n}} c^{\frac{s}{n}}} = \frac{a^{\frac{p}{m}} b^{\frac{q}{m}-\frac{r}{n}}}{c^{\frac{s}{n}}};$$

reducing the fractional exponents to the same denominator, in order to perform the subtraction, which is required, we find

$$\frac{\sqrt[m]{a^p b^q}}{\sqrt[n]{b^r c^s}} = \frac{a^{\frac{np}{mn}} b^{\frac{nq-mr}{mn}}}{c^{\frac{ms}{mn}}} = \sqrt[mn]{\frac{a^{np} b^{nq-mr}}{c^{ms}}}.$$

It is obvious, that the reduction of fractional exponents to the same denominator, answers here to the reduction of radical expressions to the same degree, and leads to precisely the same results (171).

CLXXXV. It is also very evident, by the rule given in art. 127, that

$$\left(\sqrt[m]{a^p}\right)^n = \left(a^{\frac{p}{m}}\right)^n = a^{\frac{np}{m}} = \sqrt[m]{a^{np}},$$

and by the rule laid down in art. 135, that

$$\sqrt[n]{\sqrt[m]{a^p}} = \sqrt[n]{a^{\frac{p}{m}}} = a^{\frac{p}{mn}} = \sqrt[nm]{a^p}.$$

The calculus of fractional exponents affords one of the most remarkable examples of the utility of signs, when well chosen. The analogy which prevails among exponents, both fractional and entire, renders the rules, that are to be followed with respect to the latter, applicable also to the former; but a particular investigation is necessary in each case, when we use the sign $\sqrt{}$, because it has no connexion with the operation that is indicated. The further we advance in algebra, the more fully shall we be convinced of the numerous advantages which arise from the notation by exponents, introduced by Descartes.

Examples in the Formation of Powers of Compound Algebraic Expressions.

1. $(a - b)^3 = a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3.$
2. $(4 - 3\,b)^3 = 64 - 144\,b + 108\,b^2 - 27\,b^3.$
3. $(5 - 4\,x)^4 = 625 - 2000\,x + 2400\,x^2 - 1280\,x^3 + 256\,x^4.$
4. $(a^3 + 3\,a\,b)^4 = a^{12} + 12\,a^{10}\,b + 54\,a^8\,b^2 + 108\,a^6\,b^3 + 81\,a^4\,b^4.$
5. $(5\,a^2\,c^2\,d - 4\,a\,b\,d^2)^4 = 625\,a^8\,c^8\,d^4 - 2000\,a^7\,b\,c^6\,d^5 + 2400\,a^6\,b^2\,c^4\,d^6 - 1280\,a^5\,b^3\,c^2\,d^7 + 256\,a^4\,b^4\,d^8.$
6. $(3\,a\,c - 2\,b\,d)^5 = 243\,a^5\,c^5 - 810\,a^4\,c^4\,b\,d + 1080\,a^3c^3b^2d^2 - 720\,a^2\,c^2\,b^3\,d^3 + 240\,a\,c\,b^4\,d^4 - 32\,b^5\,d^5.$
7. $(\sqrt{a} + \sqrt{b})^4 = a^2 + 6\,a\,b + b^2 + (4\,a + 4\,b)\sqrt{a\,b}.$
8. $(a + b + c)^3 = a^3 + 3\,a^2\,b + 3\,a^2\,c + 3\,a\,b^2 + 6\,a\,b\,c + 3\,a\,c^2 + b^3 + 3\,b^2\,c + 3\,b\,c^2 + c^3.$

9. $(a + 2b + c)^3 = a^3 + 6a^2b + 3a^2c + 12ab^2 + 12abc + 3ac^2 + 8b^3 + 12b^2c + 6bc^2 + c^3$.

10. $(a + b + c + d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$.

11. $(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1} b^1 + \frac{n(n-1)}{1 \times 2} a^{n-2} b^2 \pm \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3} b^3 + \ldots \ldots\ldots$

12. Make $n = \frac{1}{2}$, $n = \frac{1}{3}$, $n = \frac{1}{4}$, $n = \frac{2}{3}$, ...

Examples in the Extraction of the Cube Roots of Numbers.

1. $\sqrt[3]{12167} = 23$.
2. $\sqrt[3]{75207} = 43$.
3. $\sqrt[3]{456533} = 77$.
4. $\sqrt[3]{884736} = 96$.
5. $\sqrt[3]{1191016} = 106$.
6. $\sqrt[3]{2460375} = 135$.
7. $\sqrt[3]{347428927} = 703$.
8. $\sqrt[3]{12} = 2.28942$ .. ...
9. $\sqrt[3]{5.8} = 1.79670$ ... ..
10. $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$.
11. $\sqrt[3]{\frac{27}{64}} = \frac{3}{4}$.
12. $\sqrt[3]{\frac{2}{3}} = 0.87358$ .. ..
13. $\sqrt[3]{\frac{5}{6}} = 0.94103$ ......
14. $\sqrt[3]{3\frac{4}{5}} = 1.56049$ .....

Examples in the Extraction of the Cube Roots of Algebraic Expressions.

1. $\sqrt[3]{(6x^2 + x^3 + 8 + 12x)} = x + 2$.
2. $\sqrt[3]{(x^6 - 6cx^5 + 12c^2x^4 - 8c^3x^3)} = x^2 - 2cx$.
3. $\sqrt[3]{(a^{2m} - 6a^{2m+1}x^n + 12a^{m+2}x^{2n} - 8a^3x^{3n})} = a^m - 2ax^n$.

4. $\sqrt[3]{(a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3)} = a + b + c.$

5. $\sqrt[3]{\left(8a^3 - 4a^2b + 12a^2c^2 + \frac{3ab^2}{2} - 6abc^2 + 6ac^4 - \frac{b^3}{8} + \frac{3b^2c^2}{4} - \frac{3bc^4}{2} + c^6\right)} = 2a - \frac{b}{2} + c^2.$

6. $\sqrt[3]{(a^3 - x^3)} = a - \frac{x^3}{3a^2} - \frac{x^6}{9a^5} - \frac{5a^9}{81a^8} - \frac{10a^{12}}{243a^{11}} + \dots\dots$

7. $\sqrt[3]{(a^3 + x^3)} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} + \dots\dots$

8. $\sqrt[3]{(1 - x)} = 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \dots\dots$

9. $\sqrt[3]{(1 + x)} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{213} + \dots\dots$

Examples in the Multiplication of Radical Expressions.

1. $a\sqrt[n]{x} \times b\sqrt[n]{y} \times c\sqrt[n]{z} = abc\sqrt[n]{xyz}.$
2. $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5} = \frac{7}{2}\sqrt[3]{120}.$
3. $4 \times 2\sqrt[6]{3} \times \sqrt[6]{72} = 8\sqrt{6}.$
4. $5\sqrt{3} \times 7\sqrt{\frac{8}{7}} \times \sqrt{2} = 140.$
5. $c\sqrt{a} \times d\sqrt{a} = acd$
6. $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{5} = \sqrt[12]{648000}$
7. $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{3}} \times \sqrt[8]{3} = \sqrt[24]{\frac{256}{3}}.$
8. $\sqrt[12]{\frac{a}{bc}} \times \sqrt[8]{\frac{a^m}{b}} = \sqrt[24]{\frac{a^{3m+2}}{b^5c^2}}.$
9. $\frac{ac}{b^3d^3}\sqrt[3]{\frac{bcd}{e}} \times \sqrt[6]{\frac{b^{10}de}{a^2c^2}} = \frac{1}{bd^2}\sqrt[6]{\frac{a^4c^6}{d^3e}}.$

10. $(\sqrt{5} + 2\sqrt{7} + 3\sqrt{10}) \times 2\sqrt{5} = 10 + 4\sqrt{35} + 30\sqrt{2}$.

11. $(3 + \sqrt{5}) \times (2 - \sqrt{5}) = 1 - \sqrt{5}$.

12. $(7 + 2\sqrt{6}) \times (9 - 5\sqrt{6}) = 3 - 17\sqrt{6}$.

13. $(-5 - \sqrt{\tfrac{3}{4}}) \times (-5 + \sqrt{\tfrac{3}{4}}) = 24\tfrac{1}{4}$.

14. $(9 + 2\sqrt{10}) \times (9 - 2\sqrt{10}) = 41$.

15. $(\sqrt[3]{5} - 2\sqrt[3]{6}) \times (3\sqrt[3]{4} - \sqrt[3]{36}) = 12 + 3\sqrt[3]{20} - 6\sqrt[3]{24} - \sqrt[3]{180}$.

16. $(2\sqrt{3} + \sqrt[3]{2}) \times (2 + \sqrt[3]{9}) = 4\sqrt{3} + 2\sqrt[3]{2} + \sqrt[3]{18} + 6\sqrt{3}$.

17. $(\sqrt{a} - \sqrt{b}) \times (\sqrt{a} + \sqrt{b}) = a - b$.

18. $(c\sqrt{a} + d\sqrt{b}) \times (c\sqrt{a} - d\sqrt{b}) = a c^2 - b d^2$

19. $\left(\sqrt{\frac{a d^2}{c^2}} + \sqrt{\frac{a^2}{b}}\right) \times (\sqrt{a c} + \sqrt{b^3}) = \frac{ad}{c} + a b + \left(a + \frac{b^2 d}{c^2}\right)\sqrt{\frac{a c}{b}}$.

20. $(\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c})^2 = \sqrt{a} + \sqrt{b} + \sqrt{c} + 2\sqrt[4]{a b} + 2\sqrt[4]{a c} + 2\sqrt[4]{b c}$.

Examples in the Division of Radical Expressions.

1. $c\sqrt[m]{a} \div d\sqrt[m]{b} = \frac{c}{d}\sqrt[m]{\frac{a}{b}}$.

2. $2 a b^2 c^2 \div 4\sqrt[3]{a^2 b c^5 d} = \frac{1}{2}\sqrt[3]{\frac{b^5 c^4}{d}}$.

3. $\sqrt[3]{a^2 b c} \div \sqrt{a b^2 c^3} = \sqrt[15]{\frac{a^7}{b c^4}}$.

4. $4\sqrt[3]{12} \div 2\sqrt{3} = 2\sqrt[6]{\tfrac{16}{3}}$.

5. $1 \div (\sqrt{3} + 2) = 2 - \sqrt{3}$.

6. $(1 + \sqrt{2}) \div (2 - \sqrt{2}) = 2 + \tfrac{3}{2}\sqrt{2}$.

7. $(5 - 7\sqrt{3}) \div (1 + \sqrt{3}) = 6\sqrt{3} - 13$.

8. $(6 - 3\sqrt{5}) \div (\sqrt{5} - 1) = \frac{3}{4}\sqrt{5} - \frac{9}{4}.$

9. $(\sqrt{3} + \sqrt{2}) \div (\sqrt{3} - \sqrt{2}) = 5 + 2\sqrt{6}.$

10. $1 \div (\sqrt{2} + \sqrt{3} - \sqrt{5}) = \frac{\sqrt{30}}{12} + \frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{6}.$

11. $\left[a^2 + b^2 + c^2 + 2(ab + ac + bc) + 2(a + b + c)(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})\right] \div (a + b + c) = (\sqrt{a} + \sqrt{b} + \sqrt{c})^2.$

Examples in the Calculus of Fractional Exponents.

(a) Multiplication.

1. $a^{\frac{3}{4}} \times a^{\frac{5}{3}} = a^{\frac{29}{12}} = a^2 \sqrt[12]{a^5}.$

2. $a^{-\frac{1}{2}} \times a^{\frac{7}{4}} \times a^{-\frac{1}{5}} = a^{\frac{21}{20}} = a \sqrt[20]{a}.$

3. $a^{-\frac{3}{4}} \times a^{-\frac{7}{8}} = a^{-\frac{13}{8}} = \frac{1}{a \sqrt[8]{a^5}}$

4. $a^{-\frac{3}{4}} b^{-2} \times a^{\frac{5}{8}} b^{\frac{1}{2}} c = a^{\frac{1}{12}} b^{-\frac{3}{2}} c = -\frac{c}{b} \sqrt[12]{\frac{a}{b^6}}.$

5. $\sqrt[5]{a^{12}} \times \sqrt[7]{a^3} \times \sqrt[6]{a^4} = a^{\frac{12}{5}} \cdot a^{\frac{3}{7}} \cdot a^{\frac{2}{5}} = a^{\frac{367}{105}}$

$= a^3 \sqrt[105]{a^{52}}.$

6. $\sqrt[5]{\frac{(c^2 - y^2)^8}{(a + x)^8}} \times \sqrt[6]{\frac{(c^2 - y^2)^{\frac{3}{2}}}{a + x}} = (c^2 - y^2)^{\frac{17}{20}}$

$\times (a + x)^{-\frac{53}{30}} = \frac{c^2 - y^2}{(a + x)^2} \sqrt[60]{\frac{(a + x)^{14}}{(c^2 - y^2)^9}}.$

7. $(\sqrt[4]{a^3} + \sqrt[5]{b^4}) \times (\sqrt[4]{a^3} - \sqrt[5]{b^2}) = (a^{\frac{3}{4}} + b^{\frac{2}{5}})$

$\times (a^{\frac{3}{4}} - b^{\frac{2}{5}}) = a^{\frac{5}{2}} - b^{\frac{4}{5}} = a\sqrt{a} - \sqrt[5]{b^4}.$

(b) *Division.*

1. $a^{-\frac{m}{n}} \div a^{\frac{p}{q}} = a^{-\frac{m}{n}-\frac{p}{q}} = a^{-\frac{mq+np}{nq}}.$

2. $c\,a^{\frac{3}{4}} \div d\,a^{\frac{5}{6}} = \dfrac{c\,a^{-\frac{1}{12}}}{d} = \dfrac{c}{d\sqrt[12]{a}}.$

3. $a^{\frac{3}{5}}\,b^{\frac{1}{2}} \div a^{-\frac{7}{5}}\,b^{-\frac{1}{4}}\,c = \dfrac{a^2\,b^{\frac{3}{4}}}{c} = \dfrac{a^2}{c}\sqrt[4]{b^3}.$

4. $\dfrac{a^{-\frac{9}{2}}\,b^{\frac{2}{3}}}{e^{\frac{14}{2}}\,c^{\frac{1}{8}}\,d^{\frac{10}{3}}} \div \dfrac{a^{-\frac{29}{4}}\,d^{-\frac{11}{3}}}{b^{\frac{8}{5}}\,c} = \dfrac{a^8\,b^2\,c}{e^7}\sqrt[60]{\dfrac{b^{16}\,d^{20}}{a^{15}\,c^{10}}}$

$$= \frac{a^3\,b^2\,c}{e^7}\sqrt[12]{\frac{d^4}{a^3\,c^4}} \times \sqrt[15]{b^4}.$$

5. $(\sqrt[4]{a^3} - 4\,b^3) \div (\sqrt[4]{a} - \sqrt[4]{b}) = a^{\frac{1}{2}} + a^{\frac{1}{4}}\,b^{\frac{1}{4}}$

$$+ b^{\frac{1}{2}} = \sqrt{a} + \sqrt{b} + \sqrt[4]{ab}.$$

(c) *Powers of Powers.*

1. $\left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}} = \dfrac{1}{\sqrt[q]{\sqrt[r]{a^{mp}}}} = a^{-\frac{mp}{nq}} = \dfrac{1}{\sqrt[nq]{a^{mp}}}.$

2. $\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}} = \dfrac{1}{\sqrt[q]{\left(\dfrac{1}{\sqrt[n]{a^m}}\right)^p}} = a^{\frac{mp}{nq}} = \sqrt[nq]{a^{mp}}.$

3. $(a^{\frac{5}{4}}\,b^{\frac{2}{3}})^{\frac{1}{3}} = \sqrt[36]{a^{15}\,b^8}.$

4. $\left(a^4\,b^{-\frac{1}{2}}\,c^{\frac{2}{5}}\right)^{-\frac{1}{4}} = \sqrt[40]{\dfrac{b^5}{a^{40}\,c^4}}.$

5. $\sqrt[6]{\left(a^3\,b\sqrt[5]{a^3\,b\,c}\right)^5} = a^3\,b\sqrt[6]{c}.$

6. $\left(\dfrac{c^2\,d}{(a+b)^{\frac{3}{2}}}\right)^{-\frac{1}{3}} = \sqrt[6]{\dfrac{(a+b)^3}{c^4\,d^2}}.$

7. $\sqrt[4]{\left(\dfrac{a\sqrt{b}}{\sqrt[3]{a\,b}}\right)^3} = \sqrt[8]{a^4\,b}.$

General Theory of Equations

CLXXXVI. Equations of the first and second degree are, properly speaking, the only ones, which admit of a complete solution; but there are general properties of equations of whatever degree, by which we are able to solve them, when they are numerical, and which lead to many conclusions, of use in the higher parts of algebra. These properties relate to the particular form, which every equation is capable of assuming.

An equation in its most general form must contain all the powers of the unknown quantity, from that of the degree of the equation to the first degree, multiplied each by some known quantity, together with one term wholly known.

A general equation of the fifth degree, for example, contains all the powers of the unknown quantity, from the first to the fifth; and if there are several terms involving the same power of the unknown quantity, we must suppose them to be united in one; according to the method given for equations of the second degree, art. 112. All the terms of the equation are then to be brought into one member, as in the article above referred to; the other member will necessarily be zero; and when the first term is negative, it is rendered positive by changing the signs of all the terms of the equation.

In this way we obtain an expression similar to the following;

$$n x^5 + p x^4 + q x^3 + r x^2 + s x + t = 0,$$

in which it is to be observed, that the letters n, p, q, r, s, t, may represent negative as well as positive numbers; then dividing the whole by n, in order that the first term may have only unity for its coefficient, and making

$$\frac{p}{n} = P,\ \frac{q}{n} = Q,\ \frac{r}{n} = R,\ \frac{s}{n} = S,\ \frac{t}{n} = T,$$

we have

$$x^5 + P x^4 + Q x^3 + R x^2 + S x + T = 0.$$

In future, we shall suppose, that equations have always been prepared as above, and shall represent the general equation of any degree whatever by

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0.$$

The interval denoted by the points may be filled up, when the exponent n takes a determinate value.

Every quantity or expression, whether real or imaginary, which, put in the place of the unknown quantity x in an equa-

tion prepared as above, renders the first member equal to zero, and which consequently satisfies the question, is called the *root of the proposed equation;* but as the inquiry does not at present relate to powers, this acceptation of the term *root* is more general, than that, in which it has hitherto been used (90, 137).

CLXXXVII. Take a proposition analogous to those given in articles 116 and 167, and one which may be regarded as fundamental.

If the root of any equation whatever,

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0,$$

be represented by a, *the first member of this equation may be exactly divided by* x — a.

Indeed, since a is one value of x, we have necessarily,

$$a^n + P a^{n-1} + Q a^{n-2} \ldots\ldots + T a + U = 0,$$

and, consequently,

$$U = - a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a,$$

so that the equation proposed is precisely the same as

$$\left.\begin{array}{l} x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x \\ - a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a \end{array}\right\} = 0,$$

which may be reduced to

$$\left.\begin{array}{r} x^n - a^n + P (x^{n-1} - a^{n-1}) + Q (x^{n-2} - a^{n-2}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + T (x - a) \end{array}\right\} = 0.$$

As the quantities

$$x^n - a^n,\ x^{n-1} - a^{n-1},\ x^{n-2} - a^{n-2},\ \ldots\ldots\ldots\ x - a,$$

are each divisible by $x - a$ (166), it is evident, that the first member of the proposed equation is made up of terms, all of which are divisible by this quantity, and may consequently be divided by $x - a$, as the enunciation of the proposition requires*.

* D'Alembert has proved the same proposition in the following manner.

If we conceive the first member of the proposed equation to be divided by $x - a$, and the operation continued until all the terms involving x are exhausted, the remainder, if there be any, cannot contain x. If we represent this remainder by R, and the quotient to which we arrive, by Q, we have necessarily

$$x^n + P x^{n-1} \ldots\ldots + \&c. = Q (x - a) + R.$$

Now if we substitute a in the place of x, the first member is reduced to nothing, since a is the value of x; the term $Q (x - a)$ is also nothing, because the factor $x - a$ becomes zero; we must, therefore, have $R = 0$, and it is so, independently of the substitution of a; for, as this remainder does not contain x, the substitution cannot take place, and it still preserves the value it had before.

Hence it follows, that in every case, $R = 0$, and that, consequently,

$$x^n + P x^{n-1} + Q x^{n-2} \ \&c.$$

is exactly divisible by $x - a$.

CLXXXVIII. To form the quotient we have only to substitute for the quantities

$$x^n - a^n,\ x^{n-1} - a^{n-1},\ x^{n-2} - a^{n-2},\ \ldots\ldots\ x - a,$$

the quotients, which are obtained by dividing these quantities by $x - a$, and which are respectively

$$\begin{array}{r} x^{n-1} + a\,x^{n-2} + a^2\,x^{n-3} \ldots\ldots + a^{n-1}, \\ x^{n-2} + a\ x^{n-3} \ldots\ldots + a^{n-2}, \\ x^{n-3} \ldots\ldots + a^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ + 1. \end{array}$$

Arranging the result with reference to the powers of x, we have

$$\begin{array}{r} x^{n-1} + \quad a\,x^{n-2} + \quad a^2\,x^{n-3} \ldots\ldots + \quad a^{n-1}, \\ + P\,a\,x^{n-2} + P\,a\ x^{n-3} \ldots\ldots + P\,a^{n-2}, \\ + Q\quad x^{n-3} \ldots\ldots + Q\,a^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T. \end{array}$$

CLXXXIX. It is evident from the rules of division alone, that if the first member of the equation,

$$x^n + P\,x^{n-1} + Q\,x^{n-2} + \&c. = 0,$$

be divided by $x - a$, the quotient obtained will be exhibited under the following form,

$$x^{n-1} + P'\,x^{n-2} + Q'\,x^{n-3} + \&c.$$

P', Q', &c. representing known quantities different from P, Q, &c. we have then

$$x^n + P\,n^{n-1} + \&c. = (x - a)\ (x^{n-1} + P'\,x^{n-2} + \&c.)\,;$$

and according to what was observed in art. 120, the proposed equation may be verified in two ways, namely, by making

$$x - a = 0, \quad \text{or} \quad x^{n-1} + P'\,x^{n-2} + \&c. = 0.$$

Now if the equation

$$x^{n-1} + P'\,x^{n-2} + \&c. = 0$$

has a root b, its first member will be divisible by $x - b$; we have then

$$x^{n-1} + P'\,x^{n-2} + \&c. = (x - b)\ (x^{n-2} + P''\,x^{n-3} + \&c.),$$

and, consequently,

$$x^n + P\,x^{n-1} + \&c. = (x - a)\ (x - b)\ (x^{n-2} + P''\,x^{n-3} + \&c.)\,;$$

the equation proposed may, therefore, be verified in three ways, namely, by making

$x - a = 0$, or $x - b = 0$, or $x^{n-2} + P'' x^{n-3} +$ &c. $= 0$.

If the last of these equations has a root, c, its first member may still be decomposed into two factors,

$$(x - c)\ (x^{n-3} + P''' x^{n-4} + \&c.) = 0\ ;$$

we then have

$$x^n + P x^{n-1} + \&c.$$
$$= (x - a)\ (x - b)\ (x - c)\ (x^{n-3} + P''' x^{n-4} + \&c.)\ ;$$

from which it is obvious, that the proposed equation may be verified in four ways, namely, by making

$x - a = 0$, $x - b = 0$, $x - c = 0$, $x^{n-3} + P''' x^{n-4} +$ &c. $= 0$.

Pursuing the same reasoning, we obtain successively factors of the degrees

$$n - 4,\ n - 5,\ n - 6,\ \&c.\ ;$$

and if each of these factors, being put equal to zero, is susceptible of a root, the first member of the proposed equation is reduced to the form

$$(x - a)\ (x - b)\ (x - c)\ (x - d)\ \ldots\ldots\ (x - l)\ ;$$

that is, it is decomposed into as many factors of the first degree, as there are units in the exponent n, which denotes the degree of the equation.

The equation

$$x^n + P x^{n-1} + \&c. = 0,$$

may be verified in n ways; namely, by making

$x - a = 0$, or $x - b = 0$, or $x - c = 0$, or $x - d = 0$, or lastly, $x - l = 0$.

It is necessary to observe, that these equations are to be regarded as true only when taken one after the other, and there arise manifest contradictions from the supposition, that they are true at the same time. In fact, from the equation $x - a = 0$, we obtain $x = a$, while $x - b = 0$ gives $x = b$, results, which are inconsistent, when a and b are unequal quantities.

CXC. If the first member of the proposed equation,

$$x^n + P x^{n-1} + \&c. = 0,$$

be decomposed into n factors of the first degree,

$$x - a,\ x - b,\ x - c,\ x - d,\ \ldots\ldots\ x - l,$$

it cannot be divided by any other expression of this degree. Indeed, if it were possible to divide it by a binomial $x - \alpha$, different from the former ones, we should have

$$x^n + P\,x^{n-1} + \&c. = (x - \alpha)\;(x^{n-1} + p\,x^{n-2} + \&c.)$$

and, consequently,

$$(x - a)\;(x - b)\;(x - c)\;(x - d)\;\ldots\ldots\;(x - l)$$
$$= (x - \alpha)\;(x^{n-1} + p\,x^{n-2} + \&c.)\,;$$

now by changing x into α this becomes

$$(\alpha - a)\;(\alpha - b)\;(\alpha - c)\;(\alpha - d)\;\ldots\ldots\;(\alpha - l)$$
$$= (\alpha - \alpha)\;(\alpha^{n-1} + p\,\alpha^{n-2} + \&c.)\,;$$

The second member vanishes by means of the factor $\alpha - \alpha$, which is nothing; this is not the case with respect to the first, which is the product of factors, all of which are different from zero, so long as α differs from the several roots $a, b, c, d, \ldots l$. The supposition we have made then is not true; therefore, *an equation of any degree whatever does not admit of more binomial divisors of the first degree, than there are units in the exponent denoting its degree, and consequently cannot have a greater number of roots*.*

CXCI. An equation regarded as the product of a number of factors,

$$x - a,\; x - b,\; x - c,\; x - d,\; \&c.,$$

equal to the exponent of its degree, may take the form of the product exhibited in art. 143, with this modification, that the terms will be alternately positive and negative.

If we take four factors, for example, we have

$$\begin{array}{llll} x^4 - a\,x^3 & + a\,b\,x^2 & - a\,b\,c\,x & + a\,b\,c\,d = 0 \\ \quad\;\; - b\,x^3 & + a\,c\,x^2 & - a\,b\,d\,x & \\ \quad\;\; - c\,x^3 & + a\,d\,x^2 & - a\,c\,d\,x & \\ \quad\;\; - d\,x^3 & + b\,c\,x^2 & - b\,c\,d\,x & \\ & + b\,d\,x^2 & & \\ & + c\,d\,x^2. & & \end{array}$$

The second terms of the binomials $x - a$, $x - b$, $x - c$, &c. being the roots of the equation, taken with the contrary

---

* It ought to be remarked here, that because the binomial $x - \alpha$ is *prime* with the factors $x - a$, $x - b$, &c. it does not divide their product. Referring to art. 97, it is obvious, that *every quantity* C *which divides the product of two other quantities* A *and* B, *being prime with the one, must necessarily divide the other.* For example

Be $A = 9$, $B = 12$, $C = 4$, C is here prime with the quantity A, and it divides their product $AB = 108$, it must therefore divide the other B;

or $\frac{B}{C} = \frac{12}{4} = 3$.

sign, the properties enumerated in art. 143, and proved generally in art. 144, will, in the present case, be as follows,

The coefficient of the second term, taken with the contrary sign, will be the sum of the roots ;

The coefficient of the third term, will be the sum of the products of the roots, taken two and two ;

The coefficient of the fourth term, taken with the contrary sign, will be the sum of the products of the roots, multiplied three and three, and so on, the signs of the coefficients of the even terms being changed ;

The last term, subject also to this law, *will be the product of all the roots.*

Making, for example, the product of the three factors

$$x - 5,\ x + 4,\ x + 3,$$

equal to zero, we form the equation

$$x^3 + 2\,x^2 - 22\,x - 60 = 0,$$

the roots of which are

$$+ 5,\ - 4,\ - 3\,;$$

we have for their sum

$$5 - 4 - 3 = - 2\,;$$

for the sum of their products, taken two and two,

$$+ 5 \times - 4 + 5 \times - 3 - 4 \times - 3 = - 20 - 15 + 12 = - 23,$$

and for the product of the three roots,

$$+ 5 \times - 4 \times - 3 = 60.$$

In this way we form the coefficients, $2 - 23,\ - 60$, changing the signs of those for the second and fourth terms.

If we make the product of the factors

$$x - 2,\ x - 3, \text{ and } x + 5,$$

equal to zero, the equation thence arising

$$x^3 - 19\,x + 30 = 0,$$

as it has no term involving x^2, the power immediately inferior to that of the first term, *wants the second term ;* and the reason is, that the sum of the roots, which, taken in the contrary sign, forms the coefficient of this term, is here

$$2 + 3 - 5,$$

or zero, or in other words, the sum of the positive roots is equal to that of the negative*.

---

* It may be thought, that in order to discover the roots of any equation of the fourth degree

$$x^4 + p\,x^3 + q\,x^2 + r\,x + s = 0.$$

CXCII. We have proved (190), that an equation, considered as arising from the product of several simple factors, or factors of the first degree, can contain only as many of these factors, as there are units in the exponent n denoting the

it would be sufficient to compare it with the product of article 191, observing to put equal to each other the quantities by which the same power of x is multiplied; and it is in this manner that most elementary writers think to demonstrate, that *an equation of any degree whatever is the product of as many simple factors, as there are units in the exponent of its degree.* It will be seen by what follows, that the reasoning by which this is attempted to be proved, is defective. We stated the proposition with qualification in article 190, because it is necessary, in order to establish it unconditionally, to show that an equation of whatever degree has a root, real or imaginary, which is not easily done in an elementary work, and which happily is not necessary.

By forming the equations,

$$-a-b-c-d=p,$$
$$ab+ac+ad+bc+bd+cd=q,$$
$$-abc-abd-acd-bcd=r,$$
$$abcd=s,$$

in order to deduce from them the value of the letters, a, b, c, d, the roots of the proposed equation, the calculation would be very complicated, if, in the determination of the unknown quantities, a, b, c, d, we adopt the method of article 78; but if we multiply the first of the above equations by a^3, the second by a^2, the third by a, and add these three products to the fourth, member to member; we shall have

$$-a^4=pa^3+qa^2+ra+s,$$

from which we derive, by simple transposition,

$$a^4+pa^3+qa^2+ra+s=0.$$

This equation contains only a, but it is entirely similar to the one proposed. The difficulty of obtaining a, therefore, is the same as that of obtaining x.

"Thus," says Castillon (Mém. de Berlin, année 1789,) "it is shown in every work on algebra, that an equation of any degree we please, is formed of several simple binomials, but it is not so evident that an equation, formed by the multiplication of several simple binomials, can have such coefficients as we please."

If, instead of multiplying the first three equations in a, b, c, d, by a^3, a^2, and a, respectively, we multiply them by b^3, b^2, and b, or by c^3, c^2, c, or d^3, d^2, d, and add the products to the fourth equation, we shall have in the first

$$-b^4=pb^3+qb^2+rb+s,$$

in the second

$$-c^4=pc^3+qc^2+rc+s,$$

in the third

$$-d^4=pd^3+qd^2+rd+s;$$

from which it follows, that we are conducted to the same equation in the case of a, in that of b, &c. Indeed the quantities, a, b, c, d, being all dis-

degree of this equation; but if we combine these factors two and two, we form quantities of the second degree, which will also be factors of the proposed equation, the number of which will be expressed by

$$\frac{n\,(n-1)}{1\,.\,2} \quad (148).$$

For example, the first member of the equation

$$\begin{array}{l} x^4 - a\,x^3 + a\,b\,x^2 - a\,b\,c\,x + a\,b\,c\,d = 0 \\ \quad - b\,x^3 + a\,c\,x^2 - a\,b\,d\,x \\ \quad - c\,x^3 + a\,d\,x^2 - a\,c\,d\,x \\ \quad - d\,x^3 + b\,c\,x^2 - b\,c\,d\,x \\ \qquad\quad + b\,d\,x^2 \\ \qquad\quad + c\,d\,x^2 \end{array}$$

being the product of

$$(x-a) \times (x-b) \times (x-c) \times (x-d),$$

may be decomposed into factors of the second degree, in the six following ways;

$$\begin{array}{l} (x-a)\;(x-b) \times (x-c)\;(x-d) \\ (x-a)\;(x-c) \times (x-b)\;(x-d) \\ (x-a)\;(x-d) \times (x-b)\;(x-c) \\ (x-b)\;(x-c) \times (x-a)\;(x-d) \\ (x-b)\;(x-d) \times (x-a)\;(x-c) \\ (x-c)\;(x-d) \times (x-a)\;(x-b)\,; \end{array}$$

whence it appears, that an equation of the fourth degree may have six divisors of the second.

By combining the simple factors three and three, we form quantities of the third degree for divisors of the proposed equation; for an equation of the degree n the number will be

$$\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3},$$

and so on.

---

posed in the same manner in each equation, it is not to be supposed that one should be determined by a different operation from that of the others; and, in general, if in the investigation of several unknown quantities, we are obliged to employ for each the same reasonings, the same operations, and the same known quantities, all these quantities will necessarily be roots of the same equation.

Of Elimination among Equations exceeding the First Degree.

CXCIII. The rule given in art. 78, or the method pointed out in art. 84, is sufficient in all cases, for eliminating in two equations an unknown quantity, which does not exceed the first degree, whatever may be the degree of the others; and the rule of art. 78, is applicable, even when the unknown quantity is of the first degree in only one of the proposed equations.

If we have, for example, the equations

$$a x^2 + b x y + c y^2 = m^2,$$
$$x^2 + x y = n^2,$$

taking, in the second, the value of y, which will be

$$y = \frac{n^2 - x^2}{x},$$

and substituting this value and its square, in the place of y and y^2 in the first equation, we obtain a result involving only x.

CXCIV. If both of the proposed equations involved the second power of each of the two unknown quantities, the above method could be applied in resolving only one of the equations, either with respect to x or y.

Let there be, for example, the equations

$$a x^2 + b x y + c y^2 = m^2,$$
$$x^2 + y^2 = n^2;$$

the second gives

$$y = \pm \sqrt{n^2 - x^2}.$$

Substituting this value of y, and its square in the first, we obtain

$$a x^2 \pm b x \sqrt{n^2 - x^2} + c (n^2 - x^2) = m^2.$$

Our purpose appears to be answered, since we have arrived at a result, which does not involve the unknown quantity y, but we are unable to resolve the equation containing x, without reducing it to a rational form, by making the radical sign, under which the unknown quantity is found, to disappear.

It will be readily seen, that if this radical expression stood alone in one member, we might make the radical sign to disappear by raising this member to a square. Collecting together all the rational terms then in one member, by transposing the terms $\pm b x \sqrt{n^2 - x^2}$ and m^2, we have

$$a x^2 + c(n^2 - x^2) - m^2 = \mp b x \sqrt{n^2 - x^2};$$

taking the square of each member we form the equation

$$\left.\begin{array}{l} a^2x^4 + c^2(n^2 - x^2)^2 + m^4 \\ + 2acx^2(n^2 - x^2) - 2am^2x^2 - 2cm^2(n^2 - x^2) \end{array}\right\} = b^2x^2(n^2 - x^2),$$

which contains no radical expression.

The method, we have just employed for making the radical sign to disappear, deserves attention, on account of the frequent occasion we have to apply it; it consists in *insulating the quantity found under the radical sign, and then raising the two members of the proposed equation to the power denoted by the degree of this radical sign.*

CXCV. The complicated nature of this process, which increases in proportion to the number of radical expressions, added to the difficulty of resolving one of the proposed equations with reference to one of the unknown quantities, a difficulty, which is often insurmountable in the present state of algebra, has led those, who have cultivated this science, to seek a method of effecting the elimination without this; so that the resolution of the equations shall be the last of the operations required for the solution of the problem.

In order to render the operation more simple, we reduce equations with two unknown quantities to the form of equations with only one, by presenting only that, which we wish to eliminate. If we have, for example,

$$x^2 + a x y + b x = c y^2 + d y + e,$$

we bring all the terms into one member, and arrange them with reference to x; the equation then becomes

$$x^2 + (a y + b) x - c y^2 - d y - e = 0;$$

abridging this, by making

$$a y + b = P, \quad - c y^2 - d y - e = Q,$$

we have

$$x^2 + P x + Q = 0.$$

The general equation of the degree m with two unknown quantities must contain all the powers of x and y, which do not exceed this degree, as well as those products, in which the sum of the exponents of x and y does not exceed m; this equation then may be represented thus;

$$x^m + (a+by)x^{m-1} + (c+dy+ey^2)x^{m-2} + (f+gy+hy^2+ky^3)x^{m-3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+(p+qy+ry^2 \ldots + uy^{m-1})x + p' + q'y + r'y^2 \ldots + v'y^m = 0.$$

No coefficient is assigned to x^m in this equation, because we may always, by division, free any term of an equation we please, from the number, by which it is multiplied. Now if we make
$a + by = P$, $c + dy + ey^2 = Q$, $f + gy + hy^2 + ky^3 = R$,

. . . .

$p + qy \ldots\ldots + uy^{m-1} = T$, $p' + q'y \ldots\ldots + v'y^m = U$,
the above equation takes the following form,

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} \ldots\ldots + Tx + U = 0.$$

CXCVI. It should be observed, that we may immediately eliminate x in the two equations of the second degree,

$$x^2 + Px + Q = 0,\ x^2 + P'x + Q' = 0,$$

by subtracting the second from the first. This operation gives

$$(P - P')\,x + Q - Q' = 0,$$

whence

$$x = -\frac{Q - Q'}{P - P'};$$

substituting this value in one of the two proposed equations, the first for example, we find

$$\frac{(Q - Q')^2}{(P - P')^2} - \frac{P\,(Q - Q')}{P - P'} + Q = 0;$$

making the denominators to disappear, we have

$$(Q - Q')^2 - P\,(P - P')\,(Q - Q') + Q\,(P - P')^2 = 0,$$

then developing the two last terms, by writing $P - P'$ as their common factor

$$(Q - Q')^2 + (P - P')\,(PQ' - QP') = 0.$$

We have then only to substitute for P, Q, P', and Q', the particular values which answer to the case under consideration.

CXCVII. Before proceeding further, we shall show, how we may determine, whether the value of any one of the unknown quantities satisfies at the same time the two equations proposed. In order to make this more clear, we shall take a particular example; the reasoning employed will, however, be of a general nature.

Let there be the equations

$$x^3 + 3x^2y + 3xy^2 - 98 = 0 \ldots\ldots (1),$$
$$x^2 + 4xy - 2y^2 - 10 = 0 \ldots\ldots (2),$$

which we shall suppose furnished by a question, that gives $y = 3$.

In order to verify this supposition, we must substitute 3 in the place of y, in the proposed equation; we have then

$$x^3 + 9x^2 + 27x - 98 = 0 \ldots\ldots (a),$$
$$x^2 + 12x - 28 = 0 \ldots\ldots (b),$$

equations, which must present the same value of x, if that, which has been assigned to y, be correct. If the value of x be represented by α, the equation (a) and the equation (b) will according to what has been proved in art. 187, both of them be divisible by $x - \alpha$; they must, therefore, have a common divisor, of which $x - \alpha$ forms a part; and in fact, we find for this common divisor $x - 2$ (48); we have therefore $\alpha = 2$. Thus the value $y = 3$ fulfils the conditions of the question, and corresponds to $x = 2$.

If there remained any doubt, whether or not the common divisor of the equations (a) and (b) must give the value of x, we might remove it, by observing, that these equations reduce themselves to

$$(x^2 + 11x + 49)\ (x - 2) = 0,$$
$$(x + 14)\ (x - 2) = 0,$$

from which it is evident, that they are verified by putting 2 in the place of x.

CXCVIII. The method we have just explained, for finding the value of x, when that of y is known, may be employed immediately in the elimination of x.

Indeed, if we take the equations (1) and (2), and go through the process necessary for determining whether they have a common divisor involving x, instead of finding one, we arrive at a remainder, which contains only the unknown quantity y and numbers, that are given; and it is evident, that if we put in the place of y its value 3, this remainder will vanish, since by the same substitution, the equations (1) and (2) become the equations (a) and (b), which have a common divisor. Forming an equation, therefore, by taking this remainder and zero for the two members, we express the condition, which the values of y must fulfil, in order that the two given equations may admit, at the same time, of the same value for x.

The adjoining table presents the several steps of the operation relative to the equations,

$$x^3 + 3x^2y + 3xy^2 - 98 = 0,$$
$$x^2 + 4xy - 2y^2 - 10 = 0,$$

on which we have been employed in the preceding article. We find for the last divisor,

$$(9y^2 + 10)\,x - 2y^3 - 10y - 98;$$

and the remainder, being taken equal to zero, gives

$$
\begin{array}{l|l}
x^3+3x^2y+3y^2x-98 & x^2+4xy-2y^2-10 \\
\hline
-x^3-4x^2y+2y^2x+10x & x-y \\
\hline
\quad -x^2y+5y^2x+10x-98 & \\
\quad +x^2y+4y^2x-2y^3-10y & \\
\hline
\text{1st rem......}+(9y^2+10)x-2y^3-10y-98 &
\end{array}
$$

$$
\begin{array}{rl|l}
& x^2+4xy-2y^2-10 & (9y^2+10)x-2y^3-10y-98 \\
\hline
\text{or rather} & (9y^2+10)x^2+36xy^3-18y^4-110y^2-100 & (9y^2+10)x-2y^3-10y-98 \\
& \quad +40xy & \\
\hline
& -(9y^2+10)x^2+2xy^3+98x & x+38y^3+50y+98 \\
& \quad +10xy & \\
\hline
& \quad +38xy^3-18y^4-110y^2-100 & \\
& \quad +50xy & \\
& \quad +98x &
\end{array}
$$

$$
\begin{array}{ll}
\text{or rather} & (38y^3+50y+98)(9y^2+10)x-162y^6-1170y^4-2000y^2-1000 \\
& -(38y^3+50y+98)(9y^2+10)x+76y^6+480y^4+3920y^3+500y^2+5880y+9604 \\
\hline
\text{2nd rem.} & \qquad -86y^6-690y^4+3920y^3-1500y^2+5880y+8604
\end{array}
$$

Putting this remainder equal to zero, then dividing all its terms by 2, and changing the signs in order to make the first term positive, we have

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0,$$

$$43\,y^6 + 345\,y^4 - 1960\,y^3 + 750\,y^2 - 2940\,y - 4302 = 0,$$
an equation which admits, besides the value $y = 3$ given above, of all the other values of y, of which the questions proposed is susceptible.

The remainder above mentioned being destroyed, that preceding the last becomes the common divisor of the equations proposed; and being put into an equation, gives the value of x when that of y is introduced. Knowing, for example that $y = 3$ we substitute this value in the quantity

$$(9\,y^2 + 10)\,x - 2\,y^3 - 10\,y - 98\,;$$

then taking the result for one member, and zero for the other, we have the equation of the first degree

$$91\,x - 182 = 0, \quad \text{or} \quad x = 2.$$

CXCIX. The operation to which the above equations have been subjected, furnishes occasion for several important remarks, First, it may happen that the value of y reduces the remainder preceding the last to nothing: in this case, the next higher remainder, or that which involves the second power of x, becomes the common divisor of the two proposed equations. Introducing then into this the value of y, and putting it equal to zero, we have an equation of the second degree, involving only x, the two values of which will correspond to the known value of y. If this value still reduce to nothing the remainder of the second degree, we must go back to the preceding, or that into which the third power of x enters, because this, in the case under consideration, becomes the common divisor of the two proposed equations; and the value of y will correspond to the three values of x. In general, we must go back until we arrive at a remainder, which is not destroyed by substituting the value of y.

It may sometimes happen, that there is no remainder, or that the remainder contains only known quantities.

In the first case, the two equations have a common divisor independently of any determination of y; they assume then the following form,

$$P \times D = 0, \; Q \times D = 0,$$

D being the common divisor. It is evident, that we satisfy both the equations at the same time, by making in the first place $D = 0$; and this equation will enable us to determine one of the unknown quantities by means of the other, when the factor D contains both; but if it contains only given quantities and x, this unknown quantity will be determinate, and the other

will remain wholly indeterminate. With respect to the factors, which do not contain x, they are found by what is laid down in art. 50.

Next, if we make at the same time

$$P = 0,\ Q = 0,$$

we have still two equations, which will furnish solutions of the question proposed.

Let there be, for example,

$$(a\,x + b\,y - c)\,(m\,x + n\,y - d) = 0,$$
$$(a'\,x + b'\,y - c')\,(m\,x + n\,y - d) = 0;$$

by supposing, first, the second factor, common to the two equations, to be nothing, we have with respect to the unknown quantities x and y only the equation

$$m\,x + n\,y - d = 0,$$

and in this view the question will be indeterminate; but if we suppress this factor, we are furnished with the equations

$$a\,x + b\,y - c = 0,\quad a'\,x + b\,y - c' = 0,$$

or $$a\,x + b\,y = c,\quad a'\,x + b'\,y = c',$$

and in this case the question will be determinate, since we have as many equations as unknown quantities.

When the remainder contains only given quantities, the two proposed equations are contradictory; for the common divisor, by which it is shown that they may both be true at the same time, cannot exist, except by a condition which can never be fulfilled. This case corresponds to that mentioned in art. 68., equations of the first degree*.

CC. If then we have any two equations,

$$x^m + P\,x^{m-1} + Q\,x^{m-2} + R\,x^{m-3}\ .\ .\ .\ + T\,x + U = 0,$$
$$x^n + P'\,x^{n-1} + Q'\,x^{n-2} + R'\,x^{n-3}\ .\ .\ .\ + Y'\,x + Z' = 0,$$

* It will be readily perceived, by what precedes, that the problem for obtaining the final equation from two equations with two unknown quantities, is, in general, determinate; but the same final equation answers to an infinite variety of systems of equations with two unknown quantities. Reversing the process, by which the greatest common divisor of two quantities is obtained, we may form these systems at pleasure; but as this inquiry relates to what would be of little use in the elementary parts of mathematics, and would lead us into tedious details, we shall not pursue it here. Researches of this nature must be left to the sagacity of the intelligent reader, who will not fail, as occasion offers, of arriving at a satisfactory result.

where the second unknown quantity, y, is involved in the coefficients, P, Q, &c. P', Q', &c. the elimination of the unknown quantity x is effected: in seeking the greatest common divisor of their first members, we resolve them into other more simple expressions, or come to a remainder independent of x, which must be made equal to zero.

This remainder will form the *final equation* of the question proposed, if it does not contain factors foreign to this question; but it very often begins with polynomials involving y, by which the highest power of x, in the several quantities, that have been successively employed as divisors, is multiplied, and we arrive at a result more complicated than that which is sought, should be. In order to avoid being led into error with respect to the values of y arising from these factors, the idea, which first presents itself, is, to substitute immediately in the equations proposed, each of the values furnished by the equation involving y only; for all the values, which give a common divisor to these equations, necessarily belong to the question, and the others must be excluded. It will be perceived also, that the final equation will become incomplete, if we suppress in the operation any factor involving y; but all these circumstances together occasion some inconvenience in the application of the above method, and lead us to prefer the method given by Euler, which we shall explain in the following article.

CCI. Let there be the equations

$$x^3 + P\,x^2 + Q\,x + R = 0,$$
$$x^4 + P'\,x^3 + Q'\,x^3 + R'\,x + S' = 0;$$

representing by $x - \alpha$ the factor, which must be common to both, when y is determinate in a proper sense, we may consider the first as the product of $x - \alpha$ by the factor of the second degree, $x^2 + p\,x + q$, and the second as the product of $x - \alpha$ by the factor of the third degree $x^3 + p'\,x^2 + q'\,x + r'$, p and q, p', q' and r' being indeterminate coefficients. We have then

$$x^3 + P\,x^2 + Q\,x + R = (x - \alpha)\,(x^2 + p\,x + q),$$
$$x^4 + P'x^3 + Q'\,x^2 + R'x + S' = (x - \alpha)\,(x^3 + p'x^2 + q'\,x + r').$$

Exterminating the binomial $(x - \alpha)$, in the same manner as an unknown quantity of the first degree (84), we find

$$(x^3 + P\,x^2 + Q\,x + R)\,(x^3 + p'\,x^2 + q'\,x + r')$$
$$= (x^4 + P'\,x^3 + Q'\,x^2 + R\,x + S')\,(x^2 + p\,x + q);$$

a result, which must verify itself without any particular value being assigned to x; this cannot take place, however, unless

the first member be composed of the same terms as the second; we must, therefore, after performing the multiplications, which are indicated, put the coefficients belonging to each power of x in one member, respectively, equal to those belonging to the same power in the other. In this way we obtain the following equations;

$$P+p'=P'+p,\quad Rp'+Qq'+Pr'=S'+R'p+Q'q$$
$$Q+Pp'+q'=Q'+P'p+q,\quad Rq'+Qr'=S'p+R'q$$
$$R+Qp'+Pq'+r'=R'+Q'p+P'q,\quad Rr'=S'q.$$

As we have here six equations, and only five indeterminate quantities, namely, p, q, p', q', and r', all of which are of the first degree, these quantities may be exterminated; we shall thus arrive at an equation, which, involving only the quantities P, Q, R, P', Q', R', and S', will express a condition necessarily implied in the conditions of the question, and which, consequently, will be the final equation in y*.

Should this equation be identical, it follows, that the proposed equations have at least one factor of the form $x-\alpha$, whatever y may be; on the contrary, if the final equation contain only known quantities, the proposed equations are contradictory.

When the final equation takes place, we obtain the factor $x-\alpha$ by dividing the first of the proposed equations by the polynomial x^2+px+q; we find for the quotient

$$x+P-p,$$

and neglect the remainder, because it must necessarily be re-

* The method of Euler, explained here, amounts to multiplying each of the proposed equations by a factor, the coefficients of which are indeterminate, putting the products equal, and disposing the coefficients in such a manner, that the terms containing the unknown quantity destroy each other. In this form it is presented in his *Introduction to the Analysis of Infinites*. The exponent, which denotes the degree of the products, being designated by k, that of the factors is $k-m$ for the equation of the degree m, and $k-n$ for that of the degree n. The first term of each of these factors, having unity for a coefficient, the one contains $k-m$ indeterminate coefficients, and the other $k-n$. The sum of the products contains a number k of terms involving x; but it is necessary to destroy $k-1$ terms only, because that, which contains the highest power of x, vanishes of itself. It follows from this, that the whole number $2k-m-n$ of indeterminate coefficients must be equal to $k-1$, and consequently $k=m+n-1$; we must, therefore, multiply the equation of the degree m by a factor of the degree $n-1$, that of the degree n by a factor of the degree $m-1$, and put the products equal, term to term, a method similar to that given in the text. It may be observed, that this former method of Euler contains the germ of that developed by Bézout in his *Théorie des Equations Algébriques*.

duced to nothing, when we substitute in the place of y a value obtained from the final equation. Putting the above quotient equal to zero, we find

$$x = p - P,$$

and this value of x will be known, or at least will be expressed by means of y, if we substitute for p its value deduced from the equations of the first degree, formed above.

This expression assumes, in general, a fractional form, so that we have $x = \frac{M}{N}$, or $Nx - M = 0$; and it may be seen in this case, that the values of y, which would cause M and N to vanish at the same time, would verify the preceding equation independently of x; this takes place in consequence of the fact, that by means of these values, the proposed equations would acquire a common factor of a degree above the first. It would not be difficult to go back to the immediate conditions in which this circumstance is implied; but the limits we have prescribed in the present treatise, do not permit us to enter into details of this kind.

CCII. Now let there be the equations

$$x^2 + Px + Q = 0, \quad x^2 + P'x + Q' = 0;$$

the factors by which $x - \alpha$ is multiplied, will be here of the first degree, or $x + p$ and $x + p'$ simply; in this case,

$$R = 0, R' = 0, S' = 0, q = 0, q' = 0, r' = 0,$$

and we have

$$\left.\begin{aligned} P + p' &= P' + p \\ Q + Pp' &= Q' + P'p \\ Qp' &= Q'p \end{aligned}\right\} \text{ or } \left\{\begin{aligned} p - p' &= P - P' \\ P'p - Pp' &= Q - Q' \\ Q'p - Qp' &= 0. \end{aligned}\right.$$

From the first two equations we obtain

$$p = \frac{(P - P')P - (Q - Q')}{P - P'};$$

$$p' = \frac{(P - P')P' - (Q - Q')}{P - P'}.$$

Substituting these values in the third, we have

$$(P - P')Q'P - (Q - Q')Q' = (P - P')P'Q - (Q - Q')Q$$

or $\qquad (P - P')(PQ' - QP') + (Q - Q')^2 = 0.$

Now if in the equation

$$x = p - P;$$

we put in the place of p, its value found above, we have, as in (196)

$$x = -\frac{Q - Q'}{P - P'}.$$

CCIII. In order to aid the learner, we shall indicate the operations necessary for eliminating x in the two equations

$$x^3 + P x^2 + Q x + R = 0, \quad x^3 + P' x^2 + Q' x + R' = 0.$$

In this case, we have

$$S' = 0, \ r' = 0 \ (201),$$

and are furnished with these five equations;

$$\begin{aligned} P + p' &= P' + p, \\ Q + P p' + q' &= Q' + P' p + q, \\ R + Q p' + P q' &= R' + Q' p + P' q, \\ R p' + Q q' &= R' p + Q' q, \\ R q' &= R' q, \end{aligned}$$

which may take the following form,

$$\begin{aligned} p - p' &= P - P', \\ P' p - P p' + q - q' &= Q - Q', \\ Q' p - Q p' + P' q - P q' &= R - R', \\ R' p - R p' + Q' q - Q q' &= 0, \\ R' q - R q' &= 0. \end{aligned}$$

We may, by the rules given in art. 88, obtain immediately from any four of these equations, the values of the unknown quantities p, p', q and q'; but the simple form, under which the first and the last of the equations are presented, enables us to arrive at the result, by a more expeditious method. In order to abridge the expressions, we make

$$P - P' = e, \quad Q - Q' = e', \quad R - R' = e'';$$

and proceed to deduce from the first and last of the proposed equations,

$$p' = p - e, \ q' = \frac{R' q}{R};$$

then substituting these values in the three others, and making the denominator R to disappear, we have

$$\begin{aligned} (P' - P) R p + (R - R') q &= R (e' - P e) \ \ldots \ (a), \\ (Q' - Q) R p + (R P' - P R') q &= R (e'' - Q e) \ \ldots \ (b), \\ (R' - R) R p + (R Q' - Q R') q &= - R^2 e \ \ldots\ldots\ldots \ (c). \end{aligned}$$

If now we obtain, from the equations (a) and (b), the values of

p and q (88), and suppress the factor R, which will be common to the numerators and the denominator, we have

$$p = \frac{(e' - P e)\ (R P' - P R') - (R - R')\ (e'' - Q e)}{(P' - P)\ \ (R P' - P R') - (R - R')\ (Q' - Q)}.$$

$$q = \frac{(P' - P)\ (e'' - Q e)\ R - R\ (e' - P e)\ (Q' - Q)}{(P' - P)\ (R P' - P R') - (R - R')\ (Q' - Q)};$$

putting these values in the equation (c), we obtain a final equation, divisible by R, and which may be reduced to

$$\begin{aligned}&(R' - R)\ [(e' - Pe)\ (R P' - P R') - (R - R')\ (e'' - Q e)]\\ &+ (R Q' - Q R')\ [(P' - P)\ (e'' - Q e) - (e' - P e)\ (Q' - Q)]\\ &= -Re\ [(P' - P)\ (R P' - P R') - (R - R')\ (Q' - Q)];\end{aligned}$$

it only remains then to substitute for the letters e, e', e'', the quantities they represent.

CCIV. If we have the three unknown quantities x, y, and z, and are furnished with an equal number of equations, distinguished by (1), (2), and (3); in order to determine these unknown quantities, we may combine, for example, the equation (1) with (2) and with (3), to eliminate x, and then exterminate y from the two results, which are obtained. But it must be observed, that by this *successive* elimination, the three proposed equations do not concur, in the same manner, to form the final equation; the equation (1) is employed twice, while (2) and (3) are employed only once; hence the result, to which we arrive, contains a factor foreign to the question (84). Bézout, in his *Théorie des Equations*, has made use of a method, which is not subject to this inconvenience, and by which he proves, that *the degree of the final equation, resulting from the elimination among any number whatever of complete equations, containing an equal number of unknown quantities, and quantities of any degrees whatever, is equal to the product of the exponents, which denote the degree of these equations.* At present, we shall observe simply, that it is easy to verify this proposition in the case of the final equations presented in articles 202, and 203. If we suppose the proposed equations given in those articles to be complete, the unknown quantity y enters of the first degree into P and P', of the second degree into Q and Q', of the third into R and R'; hence it follows, that e will be of the first degree, e' of the second, and e'' of the third, and that of the terms of the highest degree found in the products indicated in the final equation given in art. 202, will have 4, or 2 . 2, for an exponent, and those of the final equation art. 203, will have 9 or 3 . 3.

Of Commensurable Roots and the equal Roots of Numerical Equations.

CCV. Having made known the most important properties of algebraic equations, and explained the method of eliminating the unknown quantities, when several occur, we shall proceed to the numerical resolution of equations with only one unknown quantity, that is, to the finding of their roots, when their coefficients are expressed by numbers*.

We shall begin by showing, that *when the proposed equation has only whole numbers for its coefficients, and that of its first term is unity, its real roots cannot be expressed by fractions, and consequently can be only whole numbers, or numbers, that are incommensurable.*

In order to prove this, let there be the equation

$$x^n + P x^{n-1} + Q x^{n-2} \ldots\ldots + T x + U = 0,$$

in which we substitute for x an irreducible fraction $\frac{a}{b}$; the equation then becomes

$$\frac{a^n}{b^n} + P\frac{a^{n-1}}{b^{n-1}} + Q\frac{a^{n-2}}{b^{n-2}} \ldots\ldots + T\frac{a}{b} + U = 0;$$

reducing all the terms to the same denominator, we have

$$a^n + P a^{n-1} b + Q a^{n-2} b^2 \ldots\ldots + T a b^{n-1} + U b^n = 0,$$

which is equivalent to

$$a^n + b\,(P a^{n-1} + Q a^{n-2} b \ldots. + T a b^{n-2} + U b^{n-1}) = 0.$$

The first member of this last equation consists of two entire parts, one of which is divisible by b, and the other is not (98), since it is supposed, that the fraction $\frac{a}{b}$ is reduced to its most simple form, or that a and b have no common divisor; one of these parts cannot therefore destroy the other.

CCVI. After what has been said, we shall perceive the utility of making the fractions of an equation to disappear, or of rendering its coefficients entire numbers, in such a manner, however, that the first term may have only unity for its coeffici-

* There is no general solution for degrees higher than the fourth; properly speaking, it is only that for the second degree, which can be regarded as complete. The expressions for the roots of equations of the third and fourth degree are very complicated, subject to exceptions, and less convenient in practice than those, which we are about to give.

ent. This is done *by making the unknown quantity proposed, equal to a new unknown quantity, divided by the product of all the denominators of the equation*, then reducing all the terms to the same denominator, by the method given in art. 52.

Let there be, for example, the equation

$$x^3 + \frac{a\,x^2}{m} + \frac{b\,x}{n} + \frac{c}{p} = 0\,;$$

we take $x = \frac{y}{m\,n\,p}$, and introducing this expression for x into the proposed equation, we obtain

$$\frac{y^3}{m^3\,n^3\,p^3} + \frac{a\,y^2}{m^2\,n^2\,p^2} + \frac{b\,y}{m\,n^2\,p} + \frac{c}{p} = 0\,;$$

as the divisor of the first term contains all the factors found in the other divisors, we may multiply by this divisor, and thus reduce each term to its most simple expression; we find then

$$y^3 + a\,n\,p\,y^2 + b\,m^2\,n\,p^2\,y + c\,m^3\,n^3\,p^2 = 0.$$

If the denominators, $m\;n\;p$, have common divisors, it is only necessary to divide y by the least number, which can be divided at the same time by all the denominators. As these methods of simplifying expressions will be readily perceived, we shall not stop to explain them; observing only, that if all the denominators were equal to m, it would be sufficient to make $x = \frac{y}{m}$.

The proposed equation, which would be in this case,

$$x^3 + \frac{a\,x^2}{m} + \frac{b\,x}{m} + \frac{c}{m} = 0,$$

then becomes

$$\frac{y^3}{m^3} + \frac{a\,y^2}{m^3} + \frac{b\,y}{m^2} + \frac{c}{m} = 0,$$

and we have

$$y^3 + a\,y^2 + b\,m\,y + c\,m^2 = 0.$$

It is evident, that the above operation amounts to multiplying all the roots of the proposed equations by the number m, since $x = \frac{y}{m}$ gives $y = m\,x$.

CCVII. Now since, if a be the root of the equation

$$x^n + P\,x^{n-1} + Q\,x^{n-2} \,\ldots\ldots + T\,x + U = 0,$$

we have

$$U = -a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a \text{ (187)},$$

it follows, that a is necessarily one of the divisors of the entire number U, and consequently, when this number has but few divisors, we have only to substitute them successively in the place of x, in the proposed equation, in order to determine whether or not this equation has any root among whole numbers.

If we have, for example, the equation

$$x^3 - 6x^2 + 27x - 38 = 0,$$

as the numbers

$$1, \quad 2, \quad 19, \quad 38,$$

are the only divisors of the number 38, we make trial of these, both in their positive and negative state; and we find, that the whole number $+2$ alone, satisfies the proposed equation, or that $x = 2$. We then divide the proposed equation by $x - 2$; putting the quotient equal to zero, we form the equation

$$x^2 - 4x + 19 = 0,$$

the roots of which are imaginary; and resolving this, we find that the proposed equation has three roots,

$$x = 2, \quad x = 2 + \sqrt{-15}, \quad x = 2 - \sqrt{-15}.$$

CCVIII. The method just explained, for finding the entire number, which satisfies an equation, becomes impracticable, when the last term of this equation has a great number of divisors; but the equation

$$U = -a^n - P a^{n-1} - Q a^{n-2} \ldots\ldots - T a,$$

furnishes new conditions, by means of which the operation may be very much abridged. In order to make the process more plain, we shall take, as an example, the equation

$$x^4 + P x^3 + Q x^2 + R x + S = 0.$$

The root being constantly represented by a, we have

$$a^4 + P a^3 + Q a^2 + R a + S = 0,$$
$$S = -R a - Q a^2 - P a^3 - a^4,$$

from which we obtain

$$\frac{S}{a} = -R - Q a - P a^2 - a^3.$$

It is evident from this last equation, that $\frac{S}{a}$ must be a whole number.

Bringing R into the first member, we have

$$\frac{S}{a} + R \quad - Q\,a - P\,a - a^{2};$$

abridging the expression by making $\frac{S}{a} + R = R'$, and dividing the two members of the equation

$$R' = -Q\,a - P\,a^{2} - a^{3}$$

by a, we have

$$\frac{R'}{a} = -Q \quad - P\,a \quad - a^{2},$$

whence we conclude, that $\frac{R'}{a}$ must also be a whole number.

Transposing Q, and making $\frac{R'}{a} + Q = Q'$, then dividing the two members by a, we obtain

$$\frac{Q'}{a} = -P - a,$$

whence we infer, that $\frac{Q'}{a}$ must be a whole number.

Lastly, bringing P into the first member, making $\frac{Q'}{a} + P = P'$, and dividing by a, we have

$$\frac{P'}{a} = -1.$$

Putting together the above mentioned conditions, we shall perceive that the number a will be the root of the proposed equation, if it satisfy the equations

$$\frac{S}{a} + R = R'.$$

$$\frac{R'}{a} + Q = Q',$$

$$\frac{Q'}{a} + P = P',$$

$$\frac{P'}{a} + 1 = 0,$$

in such a manner, as to make R', Q', and P' whole numbers.

Hence it follows, that in order to determine, whether one of the divisors a of the last term S can be a root of the proposed equation, we must,

1st. *Divide the last term by the divisor* a, *and add to the quotient the coefficient of the term involving* x;

2nd. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^2;

3rd. *Divide this sum by the divisor* a, *and add to the quotient the coefficient of the term involving* x^3;

4th. *Divide this sum by the divisor* a, *and add to the quotient unity, or the coefficient of the term involving* x^4; *the result will become equal to zero, if* a *is, in fact, the root.*

The rules given above are applicable, whatever be the degree of the equation; it must be observed, however, that the result will not become equal to zero, until we arrive at the first term of the proposed equation*.

CCIX. In applying these rules to a numerical example, we may conduct the operation in such a manner as to introduce the several trials with all the divisors of the last term, at the same time.

For the equation

$$x^4 - 9x^3 + 23x^2 - 20x + 15 = 0,$$

the operation is, as follows;

+ 15,	+ 5,	+ 3,	+ 1,	− 1,	− 3,	− 5,	− 15,
+ 1,	+ 3,	+ 5,	+ 15,	− 15,	− 5,	− 3,	− 1,
− 19,	− 17,	− 15,	− 5,	− 35,	− 25,	− 23,	− 21,
		− 5,	− 5,	+ 35,			
		+ 18,	+ 18,	+ 58,			
		+ 6,	+ 18,	− 58,			
		− 3,	+ 9,	− 67,			
		− 1,	+ 9,	+ 67,			
		0.					

All the divisors of the last term 15 are arranged, in the order of magnitude, both with the sign + and —, and placed in the same line; (this is the line occupied by the divisors *a*.)

* It would not be difficult to prove by means of the formula for the quotients given in art. 188, that the quantities $\frac{S}{a}, \frac{R'}{a}, \frac{Q'}{a}$ taken with the sign —; and with the order inverted, are the coefficients of the quotient arising from the polynomial

$$x^4 + Px^3 + Qx^2 + Rx + S$$

divided by $x - a$, and which is consequently,

$$x^3 - \frac{Q'}{a}x^2 - \frac{R'}{a}x - \frac{S}{a}.$$

The second line contains the equations arising from the number 15, divided successively by all its divisors; (this is the line for the quantities $\frac{S}{a}$.)

The third line is formed by adding to the numbers found in the preceding the coefficient -20, by which x is multiplied; (this is the line for the quantities $R' = \frac{S}{a} + R$.)

The fourth line contains the quotients of the several numbers in the preceding, divided by the corresponding divisors; (this is the line for the quantities $\frac{R'}{a}$.) In forming this line, we neglect all the numbers, which are not entire.

The fifth line results from the numbers, written in the preceding, added to the number 23, by which x^2 is multiplied; (this line contains the quantities Q'.)

The sixth line contains the quotients arising from the numbers in the preceding, divided by the corresponding divisors; (it comprehends the quantities $\frac{Q'}{a}$.)

The seventh line comprehends the several sums of the numbers in the preceding, added to the coefficient -9, by which x^3 is multiplied; (in this line are found the quantities $\frac{Q'}{a} + P$.)

Lastly, the eighth line is formed, by dividing the several numbers in the preceding by the corresponding divisors; (it is the line for $\frac{P}{a}$.) As we find -1 only in the column, at the head of which $+3$ stands, we conclude, that the proposed equation has only one commensurable root, namely $+3$; it is, therefore, divisible by $x - 3$*.

The divisors $+1$ and -1 may be omitted in the table, as it is easier to make trial of them, by substituting them immediately in the proposed equation.

CCX. Again, let there be, for example,

$$x^3 - 7x^2 + 36 = 0.$$

Having ascertained, that the numbers $+1$ and -1 do not satisfy this equation, we form the table subjoined, according to

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* Forming the quotient according to the preceding note, we find

$$x^3 - 6x^2 + 5x - 5.$$

the preceding rules, observing that, as the term involving x is wanting in this equation, x must be regarded as having 0 for a coefficient; we must, therefore, suppress the third line, and deduce the fourth immediately from the second.

$+36, +18, +12, +9, +6, +4, +3, +2, -2, -3, -4, -6, -9, -12, -18, -36$
$+1, +2, +3, +4, +6, +9, +12, +18, -18, -12, -9, -6, -4, -3, -2, -1$
****

$+1, +4, +9, +9, +4, +1,$
$-6, -3, +2, +2, -3, -6,$
$-1, -1, +1, -1, +1, +1,$
$0, \quad 0, \qquad\quad 0.$

We find in this example three numbers, which fulfil all the conditions, namely, $+6$, $+3$, and -2. Thus we obtain, at the same time, the three roots, which the proposed equation admits of; we conclude then, that it is the product of three simple factors, $x-6$, $x-3$, and $x+2$.

CCXI. It may be observed, that there are literal equations, which may be transformed, at once, into numerical ones.

If we have, for example,

$$y^3 + 2p\,y^2 - 33\,p^2\,y + 14\,p^3 = 0,$$

making $y = p\,x$, we obtain

$$p^3\,x^3 + 2\,p^3\,x^2 - 33\,p^3\,x + 14\,p^3 = 0,$$

a result, which is divisible by p^3, and may be reduced to

$$x^3 + 2\,x^2 - 33\,x + 14 = 0.$$

As the commensurable divisor of this last equation is $x+7$, which gives $x = -7$, we have

$$y = -7\,p.$$

The equation involving y is among those which are called *homogeneous equations*, because taken independently of the numerical coefficients, the several terms contain the same number of factors*.

*CCXII. When we have determined one of the roots of an equation, we may take for an unknown quantity the difference between this root and any one of the others; by this means we arrive at an equation of a degree inferior to that of the equation proposed, and which presents several remarkable properties.

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* For a more full account of the *commensurable divisors* of equations, the reader is referred to the third part of the *Elémens d' Algébre* of Clairaut. This geometer has treated of literal as well as numerical equations.

Let there be the general equation

$$x^m + P x^{m-1} + Q x^{m-2} + R x^{m-3} \ldots\ldots + T x + U = 0,$$

and let a, b, c, d, &c. be its roots; substituting $a + y$ in the place of x, and developing the powers, we have

$$\left.\begin{array}{l} a^m \quad + m a^{m-1} y \quad + \dfrac{m(m-1)}{2} a^{m-2} y^2 + \ldots + y^m \\ + P a^{m-1} + (m-1) P a^{m-2} y + \dfrac{(m-1)(m-2)}{2} P a^{m-3} y^2 + \ldots \\ + Q a^{m-2} + (m-2) Q a^{m-3} y + \dfrac{(m-2)(m-3)}{2} Q a^{m-4} y^2 + \ldots \\ + R a^{m-3} + (m-3) R a^{m-4} y + \dfrac{(m-3)(m-4)}{2} R a^{m-5} y^2 + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T a \quad + T y \\ + U \end{array}\right\} = 0.$$

The first column of this result, being similar to the proposed equation, vanishes of itself, since a is one of the roots of this equation; we may, therefore, suppress this column, and divide all the remaining terms by y; the equation then becomes

$$\left.\begin{array}{l} m a^{m-1} + \dfrac{m(m-1)}{2} a^{m-2} y + \ldots + y^{m-1} \\ + (m-1) P a^{m-2} + \dfrac{(m-1)(m-2)}{2} P a^{m-3} y + \ldots \\ + (m-2) Q a^{m-3} + \dfrac{(m-2)(m-3)}{2} Q a^{m-4} y + \ldots \\ + (m-3) R a^{m-4} + \dfrac{(m-3)(m-4)}{2} R a^{m-5} y + \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + T \end{array}\right\} = 0.$$

This equation has evidently for its $m - 1$ roots

$$y = b - a,\ y = c - a,\ y = d - a \ \ldots\ldots \ \text{\&c.}$$

We shall represent it by

$$A + \frac{B}{2} y + \frac{C}{2.3} y^2 \ldots\ldots + y^{m-1} = 0 \ldots\ldots (d),$$

abridging the expressions, by making

$$m a^{m-1} + (m-1) P a^{m-2} + (m-2) Q a^{m-3} \ldots\ldots + T = A$$
$$m(m-1) a^{m-2} + (m-1)(m-2) P a^{m-3} \ldots\ldots = B,$$

&c.

and we shall designate by V the expression

$$a^m + P a^{m-1} + Q a^{m-2} \ldots\ldots + T a + U.$$

CCXIII. If the proposed equation has two equal roots; if we have, for example, $a = b$, one of the values of y, namely $b - a$, becomes nothing; the equation (d) will therefore be verified, by supposing $y = 0$; but upon this supposition all the terms vanish, except the known term A; this last must, therefore, be nothing of itself; the value of a must, therefore, satisfy, at the same time, the two equations

$$V = 0 \text{ and } A = 0.$$

When the proposed equation has three roots equal to a, namely $a = b = c$, two of the roots of the equation (d) become nothing, at the same time, namely, $b - a$ and $c - a$. In this case the equation (d) will be divisible twice successively by $y - 0$ (187) or y; but this can happen, only when the coefficients A and B are nothing; the value of a must then satisfy, at the same time, the three equations

$$V = 0, \quad A = 0, \quad B = 0.$$

Pursuing the same reasoning, we shall perceive, that when the proposed equation has four equal roots, the equation (d) will have three roots equal to zero, or will be divisible three times successively by y; the coefficients A, B, and C, must then be nothing, at the same time, and consequently the value of a must satisfy at once the four equations,

$$V = 0, \quad A = 0, \quad B = 0, \quad C = 0.$$

By means of what has been said, we shall not only be able to ascertain, whether a given root is found several times among the roots of the proposed equation, but may deduce a method of determining, whether this equation has roots repeated, of which we are ignorant.

For this purpose, it may be observed, that when we have $A = 0$, or

$$m\,a^{m-1} + (m-1)\,P\,a^{m-2} + (m-2)\,Q\,a^{m-3} \;\ldots\; + T = 0,$$

we may consider a as the root of the equation

$$m\,x^{m-1} + (m-1)\,P\,x^{m-2} + (m-2)\,Q\,x^{m-3} \;\ldots\; + T = 0,$$

x representing, in this case, any unknown quantity whatever; and since a is also the root of the equation $V = 0$, or

$$x^m + P\,x^{m-1} + \&c. = 0,$$

it follows (197), that $x - a$ is a facter common to the two above mentioned equations.

Changing in the same manner a into x in the quantities, B, C, &c. the binomial $x - a$ becomes likewise a factor of the two new equations, $B = 0$, $C = 0$, &c. if the root a reduces to nothing the original quantities, B, C, &c.

What has been said with respect to the root a, may be applied to every other root, which is several times repeated; thus, by seeking, according to the method given for finding the greatest common divisor, the factors common to the equations,

$$V = 0, \quad A = 0, \quad B = 0, \quad C = 0, \text{ \&c.}$$

we shall be furnished with the equal roots of the proposed equation, in the following order;

The factors common to the first two equations only, are twice factors in the equation proposed; that is, if we find for a common divisor of $V = 0$ and $A = 0$, an expression of the form $(x = \alpha)$ $(x - 6)$, for example, the unknown quantity x will have to values equal to α, and two equal to 6, or the proposed equation will have these four factors,

$$(x - \alpha), \quad (x - \alpha), \quad (x - 6), \quad (x - 6).$$

The factors common, at the same time, to the first three of the above mentioned equations form triple factors in the proposed equation; that is, if the former are presented under the form $(x - \alpha)$ $(x - 6)$, the latter will take the form, $(x - \alpha)^3$ $(\alpha - 6)^3$. This reasoning may easily be extended to any length we please.

CCXIV. It may be remarked, that the equation $A = 0$, which, by changing a into x, becomes

$$m\, x^{m-1} + (m - 1)\, P\, x^{m-2} + (m - 2)\, Q\, x^{m-3} \ldots + T = 0,$$

is deduced immediately from the equation $V = 0$, or from the proposed equation,

$$x^m + P\, x^{m-1} + Q\, x^{m-2} \ldots + T\, x + U = 0,$$

by multiplying each term of this last by the exponent of the power of x, which it contains, and then diminishing this exponent by unity. We may remark here, that the term U, which is equivalent to $U \times x^0$, is reduced to nothing in this operation, where it is multiplied by 0. The equation $B = 0$ is obtained from $A = 0$, in the same manner as $A = 0$ is deduced from $V = 0$: $C = 0$ is obtained from $B = 0$, in the same manner as this from $A = 0$, and so on*.

* It is shewn, though very imperfectly, in most elementary treatises, that the divisor common to the two equations $V = 0$ and $A = 0$ contains equal factors raised to a power less by unity than that of the equation proposed; this may be readily inferred from what precedes; but for a demonstration of this proposition we refer the reader to the Complement of Lacroix's Algebra, where it is proved in a manner, both simple and new.

CCXV. To illustrate what has been said, by an example, we shall take the equation

$$x^5 - 13\,x^4 + 67\,x^3 - 171\,x^2 + 216\,x - 108 = 0;$$

the equation $A = 0$ becomes, in this case,

$$5\,x^4 - 52\,x^3 + 201\,x^2 - 342\,x + 216 = 0;$$

the divisor common to this and the proposed equation is

$$x^3 - 8\,x^2 + 21\,x - 18.$$

As this divisor is of the third degree, it must itself contain several factors: we must therefore seek, whether it does not contain some that are common to the equation $B = 0$, which is here

$$20\,x^3 - 156\,x^2 + 402\,x - 342 = 0.$$

We find, in fact, for a result $x - 3$; the proposed equation then has three roots equal to 3, or admits of $(x - 3)^3$ among the number of its factors. Dividing the first common divisor by $x - 3$, as many times as possible, that is, in this case twice, we obtain $x - 2$. As this divisor is common only to the proposed equation, and to the equation $A = 0$, it can enter only twice into the proposed equation. It is evident, then, that this equation is equivalent to

$$(x - 3)^3\ (x - 2)^2 = 0.$$

CCXVI. As the equation (d) gives the difference between b and the several other roots, when b is substituted for a, the differences between the root c and the others, when c is substituted for a, &c. and undergoes no change in its form by these several substitutions, retaining the coefficients belonging to the equation proposed, it may be converted into a general equation, which shall give all the differences between the several roots combined two and two. For this purpose, it is only necessary to eliminate a by means of the equation

$$a^m + Pa^{m-1} + Q\,a^{m-2} \ldots\ldots + T\,a + U = 0;$$

for the result being expressed simply by the coefficients, and exhibiting the root under consideration in no form whatever, answers alike to all the roots.

It is evident, that the final equation must be raised to the degree $m\ (m - 1)$; for its roots

$$\begin{array}{llll} a - b, & a - c, & a - d, & \&c. \\ b - a, & b - c, & b - d, & \&c. \\ c - a, & c - b, & c - d, & \&c. \end{array}$$

are equal in number to the number of arrangements, which the m letters, a, b, c, &c. admit of when taken two and two. Moreover, since the quantities

$$a - b \text{ and } b - a,\ a - c \text{ and } c - a,\ b - c \text{ and } c - b, \&c.$$

differ only in the sign, the roots of the equation are equal, when taken two and two, independently of the signs; so that if we have $y = \alpha$, we shall have, at the same time $y = -\alpha$. Hence it follows, that this equation must be made up of terms involving only even powers of the unknown quantity; for its first member must be the product of a certain number of factors of the second degree of the form

$$y^2 - \alpha^2 = (y - \alpha)\ (y + \alpha) \quad (192);$$

it will, therefore, itself be exhibited under the form

$$y^{2n} + p\,y^{2n-2} + q\,y^{2n-4} \ldots\ldots + t\,y^2 + u = 0.$$

If we put $y^2 = z$, this becomes

$$z^n + p\,z^{n-1} + q\,z^{n-2} \ldots\ldots + t\,z + u = 0;$$

and as the unknown quantity z is the square of y, its values will be the squares of the differences between the roots of the proposed equation.

It may be observed, that as the differences between the real roots of the proposed equation are necessarily real, their squares will be positive, and consequently the equation in z will have only positive roots, if the proposed equation admits of those only which are real.

Let there be, for example the equation

$$x^3 - 7\,x + 7 = 0;$$

putting $x = a + y$, we have

$$\left.\begin{array}{l} a^3 + 3\,a^2\,y + 3\,a\,y^2 + y^3 \\ -7\,a - 7\,y \\ +7 \end{array}\right\} = 0.$$

Suppressing the terms $a^3 - 7\,a + 7$, which, from their identity with the proposed equation, become nothing when united, and dividing the remainder by y, we have

$$3\,a^2 + 3\,a\,y + y^2 - 7 = 0;$$

eliminating a by means of this equation and of the equation

$$a^3 - 7\,a + 7 = 0,$$

we have

$$y^6 - 42\,y^4 + 441\,y^2 - 49 = 0;$$

putting $x = y^2$, this becomes

$$x^3 - 42\,x^2 + 441\,x - 49 = 0.$$

CCXVII. The substitution of $a + y$ in the place of x in the equation

$$x^m + P\,x^{m-1} + Q\,x^{m-2} \ldots\ldots + U = 0 \quad (212),$$

is also sometimes resorted to in order to make one of the terms of this equation to disappear. We then arrange the result with reference to the powers of y, which takes the place of the unknown quantity x, and consider a as a second unknown quantity, which is determined by putting equal to zero the coefficient of the term we wish to cancel; in this way we obtain

$$\left.\begin{array}{l} y^m + m\,a\,y^{m-1} + \dfrac{m\,(m-1)}{1\,.\,2}\,a^2\,y^{m-2} \ldots\ldots + a^m \\ \quad + \;P\;\; y^{m-1} + (m-1)\;P\,a\,y^{m-2} \ldots\ldots + P\,a^{m-1} \\ \qquad\qquad + \qquad\qquad Q\,y^{m-2} \ldots\ldots + Q\,a^{m-2} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \qquad\qquad\qquad\qquad\qquad\qquad + U \end{array}\right\} = 0.$$

If the term we would suppress be the second, or that which involves y^{m-1}, we make $m\,a + P = 0$, from which we deduce $a = -\dfrac{P}{m}$. Substituting this value in the result, there remain only the terms involving

$$y^m, \quad y^{m-2}, \quad y^{m-3}, \quad \&c.$$

Hence it follows, that *we make the second term of an equation to disappear, by substituting for the unknown quantity in this equation a new unknown quantity united with the coefficient of the second term taken with the sign contrary to that originally belonging to it, and divided by the exponent of the first term.*

Let there be, for example, the equation

$$x^3 + 6\,x^2 - 3\,x + 4 = 0;$$

we have by the rule

$$x = y - \frac{6}{3} = y - 2;$$

substituting this value, the equation becomes

$$\left.\begin{array}{r} y^3 - 6\,y^2 + 12\,y - \;\;8 \\ + 6\,y^2 - 24\,y + 24 \\ - \;\;3\,y + \;\;6 \\ + \;\;4 \end{array}\right\} = 0,$$

which may be reduced to

$$y^3 - 15\,y + 26 = 0,$$

in which the term involving y^2 does not appear. We may cause the third term, or that involving y^{m-2}, to disappear by putting equal to zero the sum of the quantities, by which it is multiplied, that is, by forming the equation

$$\frac{m\,(m-1)}{1\,.\,2}\,a^2 + (m-1)\,P\,a + Q = 0.$$

Pursuing this method, we shall readily perceive, that the fourth term will be made to vanish by means of an equation of the third degree, and so on to the last which can be made to disappear only by means of the equation

$$a^m + P\,a^{m-1} + Q\,a^{m-2} \,\ldots\ldots + U = 0,$$

perfectly similar to the equation proposed.

It is not difficult to discover the reason of this similarity. By making the last term of the equation in y equal to zero, we suppose, that one of the values of this unknown quantity is zero; and if we admit this supposition with respect to the equation $x = y + a$, it follows that $x = a$; that is, the quantity a, in this case, is necessarily one of the values of x.

CCXVIII. We have sometimes occasion to resolve equations into factors of the second and higher degrees. We cannot here explain in detail the several processes, which may be employed for this purpose; one example only will be given.

Let there be the equation

$$x^5 - 24\,x^3 + 12\,x^2 - 11\,x + 7 = 0,$$

in which it is required to determine the factors of the third degree; we shall represent one of these factors by

$$x^3 + p\,x^2 + q\,x + r,$$

the coefficients p, q, and r, being indeterminate. They must be such, that the first member of the proposed equation will be exactly divisible by the factor

$$x^3 + p\,x^2 + q\,x + r,$$

independently of any particular value of x; but in making an actual division, we meet with a remainder

$$\begin{aligned} &-(p^3 - 2\,p\,q - 24\,p + r - 12)\,x^2 \\ &-(p^2\,q - p\,r - q^2 - 24\,q + 11)\,x \\ &-(p^2\,r - q\,r - 24\,r - 7), \end{aligned}$$

an expression which must be reduced to nothing, independently of x, when we substitute for the letters, p, q, and r, the values that answer to the conditions of the question. We have then

$$\begin{aligned} p^3 - 2pq - 24p + r - 12 &= 0, \\ p^2 q - pr - q^2 + 24q + 11 &= 0, \\ p^2 r - qr - 24r - 7 &= 9. \end{aligned}$$

These three equations furnish us with the means of determining the unknown quantities, p, q, and r: and it is to a resolution of these, that the proposed question is reduced.

Of the Resolution of Numerical Equations by Approximation.

CCXIX. Having completed the investigation of commensurable divisors, we must have recourse to the methods of finding roots by approximation, which depend on the following principle:

When we arrive at two quantities, which, substituted in the place of the unknown quantity in an equation, lead to two results with contrary signs, we may infer, that one of the roots of the proposed equation lies between these two quantities, and is consequently real.

Let there be, for example, the equation

$$x^3 - 13x^2 + 7x - 1 = 0;$$

if we substitute, successively, 2 and 20 in the place of x, in the first member, instead of being reduced to zero, this member becomes, in the former case, equal to -31, and in the latter, to $+2939$; we may therefore conclude, that this equation has a real root between 2 and 20, that is, greater than 2 and less than 20.

As there will be frequent occasion to express this relation, we shall employ the signs $>$ and $<$, which algebraists have adopted to denote the inequality of two magnitudes, placing the greater of two quantities opposite the opening of the lines, and the less against the point of meeting. Thus we shall write

$x > 2$, to denote, that x is greater than 2,
$x < 20$, to denote, that x is less than 20.

Now in order to prove what has been laid down above, we may reason in the following manner. Bringing together the positive terms of the proposed equation, and also those which are negative, we have

$$x^3 + 7x - (13x^2 + 1),$$

a quantity, which will be negative, if we suppose $x = 2$, because, upon this supposition,

$$x^3 + 7x < 13x^2 + 1,$$

and which becomes positive, when we make $x = 20$, because, in this case,

$$x^3 + 7x > 13x^2 + 1.$$

Moreover, it is evident, that the quantities

$$x^3 + 7x \text{ and } 13x^2 + 1,$$

each increase, as greater and greater values are assigned to x, and that, by taking values, which approach each other very nearly, we may make the increments of the proposed quantities as small as we please. But since the first of the above quantities, which was originally less than the second, becomes greater, it is evident, that it increases more rapidly than the other, in consequence of which its deficiency is made up, and it comes at length to exceed the other; there must, therefore, be a point at which the two magnitudes are equal.

The value of x, whatever it be, which renders

$$x^3 + 7x = 13x^2 + 1,$$

and such a value has been proved to exist, gives

$$x^3 + 7x - (13x^2 + 1) = 0,$$

or

$$x^3 - 13x^2 + 7x - 1 \quad 0,$$

and must necessarily, therefore, be the root of the equation proposed.

What has been shewn with respect to the particular equation

$$x^3 - 13x^2 + 7x - 1 = 0,$$

may be affirmed of any equation whatever, the positive terms of which we shall designate by P, and the negative by N. Let a be the value of x, which leads to a negative result, and b that which leads to a positive one; these consequences can take place only upon the supposition, that by substituting the first value, we have $P < N$, and by substituting the second $P > N$; P, therefore, from being less, having become greater than N, we conclude as above, that there exists a value of x, between a and b, which gives $P = N$*.

* The above reasoning, though it may be regarded as sufficiently evident, when considered in a general view, has been developed in a manner, that will be found to be useful to those, who may wish to see the proofs given more in detail.

The statement here given seems to require, that the values assigned to x should be both positive or both negative, for if they have different signs, that which is negative produces a change in the signs of those terms of the proposed equation, which contain odd powers of the unknown quantity, and, consequently, the expression P and N are not formed in the same manner, when we substitute one value, as when we substitute

1. It is evident, that the increments of the polynomials P and N may be made as small as we please. Let

$$P = \alpha x^m + \beta x^n \ \ldots\ldots + \delta x^q,$$

m being the highest exponent of x; if we put $a + y$ in the place of x, this polynomial takes the form

$$A + By + Cy^2 \ldots\ldots + Ty^m,$$

the coefficients, $A, B, C, \ldots\ldots T$, being finite in number and having a finite value; the first term A will be the value the polynomial P assumes, when $x = a$; the remainder,

$$By + Cy^2 \ldots\ldots + Ty^m = y(B + Cy \ldots\ldots Ty^{m-1}),$$

will be the quantity, by which the same polynomial is increased when we augment by y the value $x = a$. This being admitted, if S designate the greatest of the coefficients, $B, C, \ldots\ldots T$, we have

$$B + Cy \ldots\ldots + Ty^{m-1} < S(1 + y \ldots\ldots + y^{m-1})$$

now

$$1 + y \ldots\ldots + y^{m-1} = \frac{1 - y^m}{1 - y} \quad (166):$$

therefore,

$$y(B + Cy \ldots\ldots + Ty^{m-1}) < Sy\frac{(1 - y^m)}{1 - y},$$

and, consequently, the quantity by which the polynomial P is increased, will be less than any given quantity c, if we make $\frac{Sy(1 - y\ )}{1 - y}$ less than this last quantity; this is effected by making $\frac{Sy}{1 - y} = c$, because, in this case, $y = \frac{}{S + c}$ being < 1, the quantity $\frac{Sy(1 - y\ )}{1 - y}$, equal to $\frac{Sy}{1 - y} - \frac{Sy^{m+1}}{1 - y}$, will necessarily be less than the quantity c, which is indefinitely small.

2. If we designate by h the increment of the polynomial P, and by k that of the polynomial N, the change, which will result from it in the value of their difference, will be $h - k$, and may be rendered smaller than a given quantity, by making smaller than this same quantity the increment, which is the greater of the two; we may, therefore, in the interval between $x = a$ and $x = b$, take values, which shall make the difference of the polynomials P and N, change by quantities as small as we please, and since the difference passes in this interval from positive to negative, it may be made to approach as near to zero as we choose.

the other. This difficulty vanishes if we make $x = 0$; in this case the proposed equation reduces itself to its last term, which has necessarily a sign contrary to that of the result arising from the substitution of one or the other of the above mentioned values. Let there be, for example, the equation

$$x^4 - 2x^3 - 3x^2 - 15x - 3 = 0,$$

the first member of which, when we put

$$x = -1 \quad \text{and} \quad x = 2,$$

becomes $+ 12$ and $- 45$. If we suppose $x = 0$, this member is reduced to $- 3$; substituting, therefore,

$$x = 0 \quad \text{and} \quad x = -1,$$

we arrive at two results with contrary signs; but putting $- y$ in the place of x, the proposed equation is changed to

$$y^4 + 2y^3 - 3y^2 + 15y - 3 = 0,$$

and we have

$$P = y^4 + 2y^3 + 15y, \; N = 3y^2 + 3,$$

whence*

$$P < N, \text{ when } y = 0,$$
$$P > N, \text{ when } y = 1.$$

Reasoning as before, we may conclude, that the equation in y has a real root, found between 0 and $+ 1$; whence it follows, that the root of the equation in x, lies between 0 and $- 1$, and, consequently, between $+ 2$ and $- 1$.

As every case the proposition enunciated can present, may be reduced to one or the other of those which have been examined, the truth of this proposition is sufficiently established.

CCXX. Before proceeding further, we shall observe, that *whatever be the degree of an equation, and whatever its coefficients, we may always assign a number, which, substituted for the unknown quantity will render the first term greater than the sum of all the others.* The truth of this proposition will be immediately apparent from what has been intimated of the rapidity, with which the several powers of a number greater than unity, increase (134); since the highest of these powers exceeds those below it more and more in proportion to the increased magnitude of the number employed, so that there is no limit to the excess of the first above each of the others. Observe, moreover, the method by which we may find a number that fulfils the condition required by the enunciation.

It is evident, that the case most unfavorable to the supposition is that, in which we make all the coefficients of the equa-

tion negative, and each equal to the greatest, that is, when, instead of

$$x^m + P x^{m-1} + Q x^{m-2} \ldots\ldots + T x + U = 0,$$

we take

$$x^m - S x^{m-1} - S x^{m-2} \ldots\ldots - S x - S = 0,$$

S representing the greatest of the coefficients P, Q, ... T, U. Giving to the first member of this equation the form

$$x^m - S (x^{m-1} + x^{m-2} \ldots\ldots + 1),$$

we may observe, that

$$x^{m-1} + x^{m-2} \ldots\ldots + 1 = \frac{x^m - 1}{x - 1} \quad (166);$$

the preceding expression then, may be changed into

$$x^m - \frac{S (x^m - 1)}{x - 1}, \text{ or into } x^m - \frac{S x^m}{x - 1} + \frac{S}{x - 1}.$$

If we substitute M for x, this becomes

$$M^m - \frac{S M^m}{M - 1} + \frac{S}{M - 1},$$

a quantity, which evidently becomes positive, if we make

$$M^m = \frac{S M^m}{M - 1}.$$

Now if we divide each member of this equation, by M^m, we have

$$1 = \frac{S}{M - 1} \text{ or } M = S + 1.$$

By substituting therefore for x the greatest of the coefficients found in the equation, augmented by unity, we render the first term greater than the sum of all the others.

A smaller number may be taken for M, if we wish simply to render the positive part of the equation greater than the negative; for to do this, it is only necessary to render the first term greater than the sum arising from all the others, when their coefficients are each equal, not to the greatest among all the coefficients but to the greatest of those which are negative; we have, therefore, merely to take for M this coefficient augmented by unity*.

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* In the *Résolution des équations numériques*, by Lagrange, there are formulæ, which reduce this number to narrower limits, but what has been said above, is sufficient to render the fundamental propositions for the resolution of numerical equations, independent of the consideration of infinity.

Hence it follows, that the positive roots of the proposed equation are necessarily comprehended within 0 and $S + 1$.

In the same way we may discover a limit to the negative roots; for this purpose we must substitute $-y$ for x, in the proposed equation, and render the first term positive, if it becomes negative (186). It is evident that by a transformation of this kind, the positive values of y answer to the negative values of x, and the reverse. If R be the greatest negative coëfficient after this change, $R + 1$ will form a limit to the positive values of y; consequently $-R - 1$ will form that of the negative value of x.

Lastly, if we would find for the smallest of the roots a limit approaching as near to zero as possible, we may arrive at it by substituting $\frac{1}{y}$ for x in the proposed equation, and preparing the equation in y, which is thus obtained, according to the directions given in art. 186. As the values of y are the reverse of those of x, the greatest of the first will correspond to the least of the second, and, reciprocally, the greatest of the second to the least of the first. If, therefore, $S' + 1$ represent the highest limit to the values of y, that is, if

$$y < S' + 1,$$

which gives

$$\frac{1}{x} < S' + 1,$$

we shall have successively,

$$1 < (S' + 1)\,x, \quad \frac{1}{S' + 1} < x.$$

Indeed, it is very evident, that we may, without altering the relative magnitude of two quantities separated by the sign $<$ or $>$, multiply or divide them by the same quantity, and that we may also add the same quantity to or subtract it from each side of the signs $<$ and $>$, which possess, in this respect, the same properties as the sign of equality.

CCXXI. It follows from what precedes, that *every equation of a degree denoted by an odd number has necessarily a real root affected with a sign contrary to that of its last term;* for if we take the number M such, that the sign of the quantity

$$M^m + P\,M^{m-1} + Q\,M^{m-2} \ldots\ldots + T\,M \pm U,$$

depends solely on that of its first term M^m, the exponent m being an odd number, the term M^m will have the same sign

as the number M (136). This being admitted, if the last term U has the sign $+$, and we make $x = -M$, we shall arrive at a result affected with a sign contrary to that, which the supposition of $x = 0$ would give; from which it is evident, that the proposed equation has a root between 0 and $-M$, that is, a negative root. If the last term U has the sign $-$, we make $x = +M$; the result will then have a sign contrary to that given by the supposition of $x = 0$, and in this case, the root will be found between 0 and $+M$, that is, it will be positive.

CCXXII. When the proposed equation is of a degree denoted by an even number, as the first term M^m remains positive, whatever sign we give to M, we are not, by the preceding observations, furnished with the means of proving the existence of a real root, if the last term has the sign $+$, since, whether we make $x = 0$, or $x = \pm M$, we have always a positive result. But when this term is negative, we find, by making

$$x = +M,\ x = 0,\ x = -M,$$

three results, affected respectively with the sign $+$, $-$, and $+$, and, consequently, the proposed equation has, at least, two real roots in this case, the one positive, found between M and 0, the other negative, between 0 and $-M$; therefore, *every equation of an even degree, the last term of which is negative, has at least two real roots, the one positive and the other negative.*

CCXXIII. We now proceed to the resolution of equations by approximation; and in order to render what is to be offered on this subject more clear, we shall begin with an example.

Let there be the equation

$$x^4 - 4x^3 - 3x + 27 = 0;$$

the greatest negative coefficient found in this equation being -4, it follows (220), that the greatest positive root will be less than 5. Substituting $-y$ for x, we have

$$y^4 + 4y^3 + 3y + 27 = 0;$$

and as all the terms of this result are positive, it appears, that y must be negative; whence it follows, that x is necessarily positive, and that the proposed equation can have no negative roots; its real roots are, therefore, found between 0 and $+5$.

The first method, which presents itself for reducing the limits, between which the roots are to be sought, is to suppose successively

$$x = 1,\ x = 2,\ x = 3,\ x = 4;$$

and if two of these numbers, substituted in the proposed equation, lead to results with contrary signs, they will form new limits to the roots. Now if we make

$x = 1$, the first member of the equation becomes $+ 21$,
$x = 2$ . . . . $+ 5$,
$x = 3$ . . . . $- 9$,
$x = 4$ . . . . $+ 15$;

it is evident, therefore, that this equation has two real roots, the one found between 2 and 3, and the other between 3 and 4. To approximate the first still nearer, we take the number 2,5, which occupies, the middle place between 2 and 3, the present limits of this root; making then $x = 2{,}5$, we arrive at the result

$$+ 39{,}0625 - 62{,}5 - 7{,}5 + 27 = - 3{,}9375;$$

as this result is negative, it is evident, that the root sought is between 2 and 2,5. The mean of these two numbers is 2,25; taking $x = 2{,}3$, we have the root sought within about one-tenth of its value, and shall approximate the true root very fast by the following process, given by Newton.

We make $x = 2{,}3 + y$: it is evident, that the unknown quantity y amounts only to a very small fraction, the square and higher powers of which may be neglected; we have then

$$\begin{aligned} x^4 &= (2{,}3)^4 + 4\,(2{,}3)^3\,y \\ -4\,x^3 &= -4\,(2{,}3)^3 - 12\,(2{,}3)^2\,y \\ -3\,x &= -3\,(2{,}3) - 3\,y; \end{aligned}$$

substituting these values, the proposed equation becomes

$$- 0{,}5839 - 17{,}812\,y = 0,$$

which gives

$$y = -\frac{0{,}5839}{17{,}812}.$$

Stopping at hundredths, we obtain for the result of the first operation

$$y = - 0{,}03 \text{ and } x = 2{,}3 - 0{,}03 = 2{,}27.$$

To obtain a new value of x, more exact than the preceding, we suppose $x = 2{,}27 + y'$; substituting this value in the proposed equation and neglecting all the powers of y' exceeding the first, we find

$$- 0{,}04595359 - 18{,}046468\,y' = 0,$$

whence

$$y' = -\frac{0{,}04595359}{18{,}046468} = - 0{,}0025,$$

and consequently $x = 2,2675$. We may, by pursuing this process, approximate, as nearly as we please, the true value of x.

If we seek the second root, contained between 3 and 4, by the same method, we find, stopping at the fourth decimal place,

$$x = 3,6797.$$

CCXXIV. We may ascertain the exactness of the method above explained, by seeking the limit to the values of the terms, which are neglected.

If the proposed equation were

$$x^m + P\,x^{m-1} + Q\,x^{m-2} \;\ldots\ldots + T\,x + U = 0,$$

substituting $a + y$ for x, we should have for the result the first of the equations found in art. 212, because a being not the root of the equation, but only an approximate value of x, cannot reduce to nothing the quantity

$$a^m + P\,a^{m-1} + Q\,a^{m-2} \;\ldots\ldots + T\,a + U.$$

Representing this last by V, we have instead of the equation (d) above referred to, the following

$$V + \frac{A}{1}\,y + \frac{B}{1\,.\,2}\,y^2 + \frac{C}{1\,.\,2\,.\,3}\,y^3 \;\ldots\ldots + y^m = 0\,;$$

from which we obtain

$$A\,y = -V - \frac{B}{1\,.\,2}\,y^2 - \frac{C}{1\,.\,2\,.\,3}\,y^3 \;\ldots\ldots - y^m,$$

$$y = -\frac{V}{A} - \frac{B\,y^2}{1\,.\,2\,A} - \frac{C\,y^3}{1\,.\,2\,.\,3\,A} \;\ldots\ldots - \frac{y^m}{A}.$$

Neglecting the powers of y exceeding the first, we have

$$y = -\frac{V}{A},$$

and this value differs from the real value of y by

$$-\frac{B\,y^2}{1\,.\,2\,A} - \frac{C\,y^3}{1\,.\,2\,.\,3\,A} \;\ldots\ldots = \frac{y^m}{A}.$$

If a differs from the true value of x only by a quantity less than $\frac{1}{p}\,a$, the above mentioned error becomes less than that, which would arise from putting $\frac{1}{p}\,a$ in the place of y, which would give

$$-\frac{B}{1\,.\,2\,A}\left(\frac{a}{p}\right)^2 - \frac{C}{1\,.\,2\,.\,3\,A}\left(\frac{a}{p}\right)^3 \;\ldots\ldots - \frac{1}{A}\left(\frac{a}{p}\right)^m.$$

Finding the value of this quantity, we shall be able to determine, whether it may be neglected when considered with reference to $\frac{V}{A}$, and if it be found too large, we must obtain for a a number, which approaches nearer to the true value of x.

To conclude, when we have gone through the calculation with several numbers. y, y', y'', &c. if the results thus obtained form a decreasing series, an approximation is certain.

CCXXV. The method we have employed above, is called the *Method by successive Substitutions*. Lagrange has considerably improved it*. He has remarked that by substituting only entire numbers, we may pass over several roots without perceiving them. In fact, if we have, for example, the equation

$$(x - \tfrac{1}{3})\ (x - \tfrac{1}{2})\ (x - 3)\ (x - 4) = 0,$$

by substituting for x the numbers 0, 1, 2, 3, &c. we shall pass over the roots $\frac{1}{3}$ and $\frac{1}{2}$, without discovering that they exist; for we shall have

$$(0 - \tfrac{1}{3})\ (0 - \tfrac{1}{2})\ (0 - 3)\ (0 - 4) = + \tfrac{1}{3} \times \tfrac{1}{2} \times 3 \times 4,$$
$$(1 - \tfrac{1}{3})\ (1 - \tfrac{1}{2})\ (1 - 3)\ (1 - 4) = + \tfrac{2}{3} \times \tfrac{1}{2} \times 2 \times 3,$$

results affected by the same sign. It will be readily perceived, that this circumstance takes place in consequence of the fact, that the substitution of 1 for x changes at the same time the signs of both the factors, $x - \frac{1}{3}$, and $x - \frac{1}{2}$, which passes from the negative state, in which they are when 0 is put in the place of x, to the positive; but if we substitute for x a number between $\frac{1}{3}$ and $\frac{1}{2}$, the sign of the factor $x - \frac{1}{3}$ alone will be changed, and we shall obtain a negative result.

We shall necessarily meet with such a number, if we substitute, in the place of x, numbers, which differ from each other by a quantity less than the difference between the root $\frac{1}{3}$ and $\frac{1}{2}$. If, for example, we substitute $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, &c. there will be two changes of the sign.

It may be objected to the above example, that when the fractional coefficients of an equation have been made to disappear, the equation can have for roots only either entire or irrational numbers, and not fractions; but it will be readily seen, that the irrational numbers, for which we have, in the example, substituted fractions for the purpose of simplifying the expressions, may differ from each other by a quantity less than unity.

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* See *Résolution des Equations Numériques.*

In general, the results will have the same sign, whenever the substitutions produce a change in the sign of an even number of factors*. To obviate this inconvenience we must take the numbers to be substituted, such, that the difference between the smallest limit and the greatest, will be less than the least of the differences, which can exist between the roots of the proposed equation; by this means the numbers to be substituted will necessarily fall between the successive roots, and will cause a change in the sign of one factor only. This process does not presuppose the smallest difference between the roots to be known, but requires only, that the limit, below which it cannot fall, be determined.

In order to obtain this limit, we form the equation involving the squares of the differences of the roots (216).

Let there be the equation

$$x^n + p\,x^{n-1} + q\,x^{n-2} \,\ldots\ldots + t\,x + u = 0 \;\ldots\ldots\; (D),$$

to obtain the smallest limit to the roots, we make (220) $x = \frac{1}{v}$;

we have then the equation

$$\frac{1}{v^n} + p\,\frac{1}{v^{n-1}} + q\,\frac{1}{v^{n-2}} \,\ldots\ldots + t\,\frac{1}{v} + u = 0,$$

or, reducing all the terms to the same denominator,

$$1 + p\,v + q\,v^2 \,\ldots\ldots + t\,v^{n-1} + u\,v \;= 0,$$

then disengaging v^n,

$$v^n + \frac{t}{u}\,v^{n-1} \ldots + \frac{q}{u}\,v^2 + \frac{p}{u}\,v + \frac{1}{u} = 0;$$

and if $\frac{r}{u}$ represent the greatest negative coefficient found in this equation, we shall have

$$\frac{1}{\frac{r}{u}+1} < x.$$

It is only necessary to consider here the positive limit, as this alone relates to the real roots of the proposed equation.

Knowing the limit

$$\frac{1}{\frac{r}{u}+1} = \frac{u}{r+u},$$

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* Equal roots cannot be discovered by this process, when their number is even; to find these, we must employ the method given in art. 213.

less than the square of the smallest difference between the roots of the proposed equation, we may find its square root, or at least, take the rational number next below this root; this number, which we shall designate by k, will represent the difference which must exist between the several numbers to be substituted. We thus form the two series,

$$0, +k, +2k, +3k, \&c.$$
$$-k, -2k, -3k, \&c.$$

from which we are to take only the terms, comprehended between the limits to the smallest and the greatest positive roots, and those to the smallest and the greatest negative roots of the proposed equation. Substituting these different numbers, we shall arrive at a series of results, which will show by the changes of the sign that take place, the several real roots, whether positive or negative.

CCXXVI. Let there be, for example, the equation

$$x^3 - 7x + 7 = 0,$$

from which, in art. 216, was derived the equation

$$z^3 - 42z^2 + 441z - 49 = 0;$$

making $z = \frac{1}{v}$, and, after substituting this value, arranging the result with reference to v, we have

$$v^3 - 9v^2 + \tfrac{42}{49}v - \tfrac{1}{49} = 0,$$

from which we obtain

$$v < 10, z > \tfrac{1}{10};$$

we must, therefore, take $k =$ or $< \frac{1}{\sqrt{10}}$. This condition will be fulfilled, if we make $k = \frac{1}{4}$; but it is only necessary to suppose $k = \frac{1}{3}$; for by putting 9 in the place of v in the preceding equation, we obtain a positive result, which must become greater, when a greater value is assigned to v, since the terms v^3 and $9v^2$ already destroy each other, and $\frac{42}{49}v$ exceeds $\frac{1}{49}$.

The highest limit to the positive roots of the proposed equation

$$x^3 - 7x + 7 = 0,$$

is 3, and that to the negative roots — 3; we must, therefore, substitute for x the numbers

$$0,\quad \tfrac{1}{3},\quad \tfrac{2}{3},\quad \tfrac{3}{3},\quad \tfrac{4}{3},\ \ldots\ldots\ldots\quad \frac{24}{3}$$
$$-\tfrac{1}{3},\ -\tfrac{2}{3},\ -\tfrac{3}{3},\ -\tfrac{4}{3},\ \ldots\ldots\ldots\ -\frac{24}{3}$$

We may avoid fractions by making $x = \dfrac{x'}{3}$; for in this case the differences between the several values of x' will be triple of those between the values of x, and, consequently, will exceed unity; we shall then have only to substitute, successively,

$$0,\quad 1,\quad 2,\quad 3,\ \ldots\ldots\ldots\quad 24,$$
$$-1,\ -2,\ -3,\ \ldots\ldots\ldots\ -24,$$

in the equation

$$x'^3 - 63\,x' + 189 = 0.$$

The signs of the results will be changed between $+4$ and $+5$, between $+5$ and $+6$, and between -9 and -10, so that we shall have for the positive values,

$$\left.\begin{array}{l} x' > 4 \text{ and } < 5 \\ x\ > 5 \text{ and } < 6 \end{array}\right\} \text{ whence } \left\{\begin{array}{l} x > \tfrac{4}{3} \text{ and } < \tfrac{5}{3} \\ x > \tfrac{5}{3} \text{ and } < \tfrac{6}{3} \end{array}\right.$$

and the negative value of x' will be found between -9 and -10, that of x between $-\tfrac{9}{3}$ and $-\tfrac{10}{3}$.

Knowing now the several roots of the proposed equation within $\tfrac{1}{3}$, we may approach nearer to the true value by the method explained in art. 223.

CCXXVII. The methods employed in the example given in art. 223, and in the preceding article, may be applied to an equation of any degree whatever, and will lead to values approaching the several real roots of this equation. It must be admitted, however, that the operation becomes very tedious, when the degree of the proposed equation is very elevated; but in most cases it will be unnecessary to resort to the equation (D), or rather its place may be supplied by methods, with which the study of the higher branches of analysis will make us acquainted*.

We shall observe, however, that by substituting successively the numbers 0, 1, 2, 3, &c. in the place of x, we shall often be led to suspect the existence of roots, that differ from each other

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* A very elegant method given by Lagrange for avoiding the use of the equation (D) may be found in the *Traité de la Résolution des Equations numériques*.

by a quantity less than unity. In the example upon which we have been employed, the results are

$$+7, +1, +1, +13,$$

which begin to increase after having decreased from $+7$ to $+1$. From this order being reversed it may be supposed, that between the numbers $+1$ and $+2$ there are two roots either equal or nearly equal. To verify this supposition, the unknown quantity should be multiplied. Making $x = \frac{y}{10}$, we find

$$y^3 - 700y + 7000 = 0,$$

an equation which has two positive roots, one between 13 and 14, and the other between 16 and 17.

The number of trials necessary for discovering these roots is not great; for it is only between 10 and 20, that we are to search for y; and the values of this unknown quantity being determined in whole numbers, we may find those of x within one tenth of unity.

CCXXVIII. When the coefficients in the equation proposed for resolution are very large, it will be found convenient to transform this equation into another, in which the coefficients shall be reduced to smaller numbers. If we have, for example

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0,$$

we may make $x = 10z$; the equation then becomes

$$z^4 - 8z^3 + 19{,}98z^2 - 14{,}937z + 0{,}5 = 0.$$

If we take the entire numbers, which approach nearest to the coefficients in this result, we shall have

$$z^4 - 8z^3 + 20z^2 - 15z + 0{,}5 = 0.$$

It may be readily discovered, that z has two real values, one between 0 and 1, the other between 1 and 2, whence it follows, that those of the proposed equation are between 0 and 10, and 10 and 20, but this must be regarded only as an indication to be verified, for it might happen that a trifling change in the coefficients of an equation, may render imaginary, roots which at first were found real, and vice versa.

CCXXIX. Lagrange has given to the successive substitutions a form which has this advantage, that it shows immediately what approaches we make to the true root by each of the several operations, and which does not presuppose the value to be known within one tenth.

Let a represent the entire number immediately below the root sought; to obtain this root, it will be only necessary to aug-

ment a by a fraction; we have, therefore, $x = a + \frac{1}{y}$. The equation involving y, with which we are furnished by substituting this value in the proposed equation, will necessarily have one root greater than unity; taking b to represent the entire number immediately below this root, we have for the second approximation $x = a + \frac{1}{b}$. But b having the same relation to y, which a has to x, we may, in the equation involving y, make $y = b + \frac{1}{y'}$, and y' will necessarily be greater than unity; representing by b' the entire number immediately below the root of the equation in y', we have

$$y = b + \frac{1}{b'} = \frac{b\,b' + 1}{b'};$$

substituting this value in the expression for x, we have

$$x = a + \frac{b'}{b\,b' + 1},$$

for the third approximation to x. We may find a fourth by making $y' = b' + \frac{1}{y''}$; for if b'' designate the entire number immediately below y'' we shall have

$$y' = b' + \frac{1}{b''} = \frac{b'\,b'' + 1}{b''};$$

whence

$$y = b + \frac{b''}{b'\,b'' + 1} = \frac{b\,b'\,b'' + b'' + b}{b'\,b'' + 1},$$

$$x = a + \frac{b'\,b'' + 1}{b\,b'\,b'' + b'' + b},$$

and so on.

CCXXX. We shall apply this method to the equation

$$x^3 - 7x + 7 = 0.$$

We have already seen (226), that the smallest of the positive roots of this equation is found between $\frac{4}{3}$ and $\frac{5}{3}$, that is, between 1 and 2; we make, therefore, $x = 1 + \frac{1}{y}$; we shall then have

$$y^3 - 4y^2 + 3y + 1 = 0.$$

The limit to the positive roots of this last equation is 5, and by substituting, successively, 0, 1, 2, 3, 4, in the place of y, we

immediately discover, that this equation has two roots greater than unity, one between 1 and 2, and the other between 2 and 3. Hence

$$x = 1 + \tfrac{1}{1}, \text{ and } x = 1 + \tfrac{1}{2},$$

that is,

$$x = 2, \text{ and } x = \tfrac{3}{2}.$$

These two values correspond to those, which were found above between $\frac{6}{3}$ and $\frac{7}{3}$, and between $\frac{5}{3}$ and $\frac{4}{3}$, and which differ from each other by a quantity less than unity.

In order to obtain the first, which answers to the supposition of $y = 1$, to a greater degree of exactness, we make

$$y = 1 + \frac{1}{y'},$$

we then have

$$y'^3 - 2y'^2 - y' + 1 = 0.$$

We find in this equation only one root greater than unity, and that is between 2 and 3, which gives

$$y = 1 + \tfrac{1}{2} = \tfrac{3}{2},$$

whence

$$x = 1 + \tfrac{2}{3} = \tfrac{5}{3}.$$

Again, if we suppose $y' = 2 + \dfrac{1}{y''}$; we shall be furnished with the equation

$$y''^3 - 3y''^2 - 4y'' - 1 = 0;$$

we find the value of y'' to be between 4 and 5; taking the smallest of these numbers, 4, we have

$$y' = 2 + \tfrac{1}{4},\ y = 1 + \tfrac{4}{9} = \tfrac{13}{9},\ x = 1 + \tfrac{9}{13} = \tfrac{22}{13}.$$

It would be easy to pursue this process, by making $y'' = 4 + \dfrac{1}{y'''}$, and so on.

We return now to the second value of x, which, by the first approximation, was found equal to $\frac{3}{2}$, and which answers to the supposition of $y = 2$. Making $y = 2 + \dfrac{1}{y'}$, and substituting this expression in the equation involving y, we have, after changing the signs in order to render the first term positive,

$$y'^3 + y'^2 - 2y' - 1 = 0.$$

This equation, like the corresponding one in the above operation, has only one root greater than unity, which is found between 1 and 2; taking $y' = 1$, we have

$$y = 3, \quad x = \tfrac{1}{3}.$$

Again assuming

$$y' = 1 + \frac{1}{y''},$$

we are furnished with the equation

$$y''^3 - 3\,y''^2 - 4\,y'' - 1 = 0,$$

in which y'' is found to be between 4 and 5, whence

$$y' = \tfrac{5}{4}, \quad y = \tfrac{14}{5}, \quad x = \tfrac{19}{14}.$$

We may continue the process by making $y'' = 4 + \dfrac{1}{y'''}$, and so on.

The equation $x^3 - 7\,x + 7 = 0$ has also one negative root, between -3 and -4. In order to approach it more nearly, we make $x = -3 - \dfrac{1}{y}$; which gives

$$y^3 - 20\,y^2 - 9\,y - 1 = 0,\ y > 20 \text{ and } < 21,$$

whence

$$x = -3 - \tfrac{1}{20} = -\tfrac{61}{20}.$$

To proceed further, we may suppose $y = 20 + \dfrac{1}{y'}$, &c. we shall then obtain, successively, values more and more exact.

The several equations transformed into equations in y, y', y'', &c. will have only one root greater than unity, unless two or more roots of the proposed equation are comprehended within the same limits a and $a + 1$; when this is the case, as in the above example, we shall find in some one of the equations in y, y', &c. several values greater than unity. These values will introduce the different series of equations, which show the several roots of the proposed equation, that exist within the limits a and $a + 1$.

The learner may exercise himself upon the following equation;

$$x^3 - 2\,x - 5 = 0,$$

the real root of which is between 2 and 3; we find for the entire values of y, y', &c.

$$10,\ 1,\ 1,\ 2,\ 1,\ 3,\ 1,\ 1,\ 12,\ \text{\&c.}$$

and for the approximate values of x,

$$\tfrac{2}{1},\ \tfrac{21}{10},\ \tfrac{23}{11},\ \tfrac{44}{21},\ \tfrac{111}{53},\ \tfrac{155}{74},\ \tfrac{576}{275},\ \tfrac{731}{349},\ \tfrac{1307}{624},\ \tfrac{16415}{7837}.$$

Equations of the Third Degree.

CCXXXI. Every equation of the 3rd degree may be represented by

$$y^3 + a\,y^2 + b\,y + c = 0,$$

the 2nd term will disappear by making $y = x - \frac{a}{3}$, by which means we obtain

$$x^3 + x\left(b - \frac{a^2}{3}\right) + \frac{2}{27}\,a^3 + c = 0$$

or by substituting p for $b - \frac{a^2}{3}$, and q for $\frac{2}{27}\,a^3 + c$,

we have $\quad x^3 + p\,x + q = 0 \quad (A),$

to which from every equation of the 3rd degree is reducible.

Substituting for x, two indeterminate quantities, as $u + z$, we get $x^3 = u^3 + z^3 + 3\,u\,z\,(u + z)$, the proposed equation then becomes

$$(3\,u\,z + p)\;(u + z) + u^3 + z^3 + q = 0.$$

But as the division of x into two numbers can be performed in an infinite number of ways, we may assume their product, their difference, their ratio, &c. Suppose then that the 1st factor $3\,u\,z + p = 0$, we get $u\,z = -\frac{p}{3}$, $u^3\,z^3 = \frac{-p^3}{27}$ or $u^3 = \frac{-p^3}{27\,z^3}$ or $z^3 = -\frac{p^3}{27\,u^3}$; and $u^3 + z^3 = -q$, $z^3 - \frac{p^3}{27\,z^3} = -q$,

$$\text{or, } z^6 + x^3 = \frac{p^3}{27} \text{ by substituting } t = u^3 \text{ and } t = z^3$$

we obtain

$$t^2 + t\,q = \frac{p^3}{27}. \qquad (B.)$$

The three roots of unity being $1,\ \alpha,\ \alpha^2$*,

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* Knowing one of the roots of a cube, we obtain the other two roots by division, in the present case let $z^3 - t$, be divided by $z - \sqrt[3]{t}$, we have $z = \sqrt[3]{t}\left(\frac{1 \pm \sqrt{-3}}{2}\right)$; or suppose we have $x^3 = 1$ or $x^3 - 1 = 0$.

we have

$$u = \sqrt[3]{t},\ \alpha\sqrt[3]{t},\ \alpha^2\sqrt[3]{t};\ z = \sqrt[3]{t'},\ \alpha\sqrt[3]{t'},\ \alpha^2\sqrt[3]{t'}.$$

But in order to obtain $x = u + z$, we must not add together all these values two by two, for then we should get 9 roots instead of 3. As, instead of the equation $u\,z = -\frac{p}{3}$, in employing its cube, we have tripled the number of its roots, we must therefore add but those of the values of u and z, of which the product is $-\frac{p}{3}$, viz. $\sqrt[3]{t\,t'}$, the 2nd member of the equation (B) being (seeing that $u\,z = -\sqrt[3]{t\,t'} = -\frac{p}{3}$, or $-u^3 z^3 = -t\,t' = \frac{p^3}{27}\Big) = -t\,t'$, the cube root of which is $-\frac{p}{3}$. It may be easily seen, that because $\alpha^3 = 1$, we ought to admit of the 9 combinations, but

$$x = \sqrt[3]{t} + \sqrt[3]{t'},\ x = \alpha\sqrt[3]{t} + \alpha^2\sqrt[3]{t'},\ x = \alpha^2\sqrt{t} + \alpha\sqrt[3]{t'}.$$

$$\begin{array}{r|l}
x - 1)\ x^3 - 1 & (x^2 + x + 1 \\
-x^3 + x^2 & \\
\hline
x^2 - 1 & \\
-x^2 + x & \\
\hline
x - 1 & \\
-x + 1 & \\
\hline
\cdot\ \ \cdot &
\end{array}$$

If $x^3 - 1$ and $x^2 - 1$ are each $= 0$, it is evident that $x^2 + x + 1 = 0$ or $x^2 + x = -1$. And as the two roots of this last equation are

$$-\frac{1}{2} + \frac{1}{2}\sqrt{-3} \text{ and } -\frac{1}{2} - \frac{1}{2}\sqrt{-3}$$

it follows, that the three roots of the equation $x^3 - 1 = 0$, or, which is the same thing, the three roots of unity are

$$1,\ -\frac{1}{2} + \frac{1}{2}\sqrt{-3} \text{ and } -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

And in the same manner it will be found, that the roots of the equation $x^3 + 1 = 0$, or, the three cube roots of -1, are

$$-1,\ +\frac{1}{2} + \frac{1}{2}\sqrt{-3} \text{ and } +\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$$

Substituting for a and a^2 their values $-\frac{1}{2}(1 \pm \sqrt{-3}$ (preceding note) and making

$$\left.\begin{array}{ll} s = \sqrt[3]{t} + \sqrt[3]{t'}, & d = \sqrt[3]{t} - \sqrt[3]{t'} \\ \text{we have} \quad x = s, & x = -\frac{1}{2}\left(s \pm d\sqrt{-3}\right) \end{array}\right\} (C).$$

To resolve therefore an equation of the 3rd degree (A), we must first resolve the equation (B); and knowing t and t' substitute their values in the formulæ (C). For example,

$x^3 + 6x = 7$, gives $p = 6, q = -7$, and the formula (B)

$t^2 - 7t = 8$; whence $t = \frac{7}{2} \pm \frac{9}{2}$ $\therefore t = 8$, and $t' = -1$;

the cube roots of which are, 2 and -1; therefore

$$s = 1,\ d = 3,\ x = 1 \text{ and } -\frac{1}{2}\left(1 \pm 3\sqrt{-3}\right)$$

Again, let $y^3 - 3y^2 + 12y = 4$; substituting $y = x + 1$, in order to eliminate the 2nd term, we have

$$x^3 + 9x + 6 = 0, \quad \therefore \quad p = 9,\ q = 6,$$

and, by formula (B)

$$t^2 + 6t = 27, \quad \therefore \quad t = 3,\ t' = -9, \text{ and}$$

$$s = \sqrt[3]{3} - \sqrt[3]{9} = -0{,}637835 = x,\ d = 3{,}522333$$

then $y = 0{,}362165$ or $1{,}318918 \pm 1{,}761167\sqrt{-3}$.

The equation $x^3 - 3x = 18$, gives $t^2 - 18t + 1 = 0$ $\therefore t = 9 \pm 4\sqrt{5}$, of which the cube root is

$$\frac{3}{2} \pm \frac{1}{2}\sqrt{5} \;\therefore\; s = 3,\ d = \sqrt{5};$$

lastly, $x = 3$, or $-\frac{1}{2}\left(3 \pm \sqrt{-15}\right)$.

Again, let $x^3 - 27x + 54 = 0$; thence $t^2 + 54t + 729 = 0$ or $(t + 27)^2 = 0 \therefore t = -27$, consequently $x = -6$ and 3 (a double root).

CCXXXII. If the *two roots* t *and* t' *are real*, $\sqrt[3]{t}$ and $\sqrt[3]{t'}$ are also so, as well as s and d; it follows then from formulæ (C), that *the proposed equation has but one real root.* However, if $t = t'$ we have $d = 0$, then the three values of x

are real. two being equal to the half of the 3rd, with a contrary sign, as has been seen by the last example.

But, *should the equation* (B) *have its two roots imaginary* (p is negative, and moreover $4\,p^3 > 27\,q^2$), the expressions (C) remaining encumbered with imaginary quantities, it appears that no root is real, contrary to what we know (221).

CCXXXIII. This paradox, which has resisted for a long time the efforts of the most eminent mathematicians, has received the name of the *irreducible case.* It remains therefore to be demonstrated that the *three roots are real.*

The values of t and t', being represented by $a \pm b\sqrt{-1}$, the cube root or the $\frac{1}{3}$ power, may be developed (page 251, note) in a series. Without performing that operation, it is evident, that imaginary quantities are to be found in such terms where $b\sqrt{-1}$ is affected by odd exponents; and as one of these series is deduced from the other by changing b into $-b$; it is obvious that they are both comprised in the form $P \pm Q\sqrt{-1}$, of which the sum is $s = 2\,P$ and the difference $d = 2\,Q\sqrt{-1}$. Thus the formulæ (C) may be reduced to the following rational expressions,

$$x = 2\,P, \text{ and } -P \pm Q\sqrt{3} \qquad (D).$$

CCXXXIV. Our roots are then real, precisely when the equations (C) give them under an imaginary form. This singular case proceeds from our taking $x = u + z$ and $u\,z = -\frac{p}{3}$, as nothing expresses that u and z are real; and our calculation goes to prove that they are imaginary whenever the three roots are real. In order to obtain these, we must develope $a + b\sqrt{-1}$ according to the power $\frac{1}{3}$ under the form of $P + Q\sqrt{-1}$; the values of P and Q will be found by the equations (D).

CCXXXV. We shall now introduce a much more simple process given by *Clairaut* in his *Elements of Algebra.*

The general equation $x^3 - p\,x + q = 0$ may be transformed into $z^3 - z = r$, by taking $x = m\,z$, and so that the coefficient of z may be equal to unity.

By this substitution we get

$$z^3 - \frac{p\,z}{m^2} = -\frac{q}{m^3}$$

taking $m^2 = p$, we have

$$z^3 - z = -\frac{q}{m^3}$$

and $m = \pm\sqrt{p}$, $r = -\frac{q}{m^3}$;

to get the last of these values always positive, m must be taken with a sign contrary to that of q. This being premised, the equation $z^3 - z = r$, can only fall within the irreducible case, when $\frac{1}{27} > \frac{r^2}{4}$, viz. when $r < \sqrt{\frac{4}{27}}$, or $< \frac{2}{3\sqrt{3}}$, which can only take place when the positive value of z lies between 1 and $\frac{2}{\sqrt{3}}$. It is evident, that z must exceed unity in order that the quantity $z^3 - z$ be positive; but by making $z = \frac{2}{\sqrt{3}} = \sqrt{\frac{4}{3}}$ the result would be $\frac{2}{3\sqrt{3}}$, which number is greater than r.

Supposing $z = 1 + \delta$, then δ can only represent a fraction, less than 0,1547 which is the difference between 1 and $\sqrt{\frac{4}{3}}$; the cube of this fraction $= 0{,}0037$ may be omitted, and at the substitution of $1 + \delta$ for z in the proposed equation, omitting δ^3, gives

$$2\,\delta + 3\,\delta^2 = r,$$

we obtain

$$\delta = \frac{-1 \pm \sqrt{1 + 3\,r}}{3}$$

and consequently

$$z = 1 + \delta = \frac{2 + \sqrt{1 + 3r}}{3}$$

the value of z, which exceeds unity, being alone required. The greatest error which can possibly be committed by this method can only amount to the thousandth part of unity. Indeed if we suppose $z = \sqrt{\frac{4}{3}}$, which value corresponds to $r = \frac{1}{3}\sqrt{\frac{4}{3}}$,

and for which δ is the greatest possible, the above formula gives

$$z = \frac{2 + \sqrt{1 + \sqrt{\frac{4}{3}}}}{3} \text{ instead of } \sqrt{\frac{1}{3}},$$

which only differs by 0,00126.

In the equation, for example

$$x^3 - 13x + 5 = 0$$

making $x = -z\sqrt{13}$, we get

$$z^3 - z = \frac{5}{13\sqrt{13}}$$

whence we deduce

$$r = \frac{5}{13\sqrt{13}},\ z = \frac{2 + \sqrt{1 + \frac{15}{13\sqrt{13}}}}{3}, \text{ and}$$

$$x = \frac{-2\sqrt{13} - \sqrt{13 + \frac{15}{\sqrt{13}}}}{3} = -3{,}784.$$

If greater exactness were required, we might employ the method given in art. (223) by which we find

$$x = -3{,}78434.$$

CCXXXVI. Let us now proceed to explain the method of solving equations of the 3rd degree by means of *the tables of Sines, Tangents,* &c.

The general formula, to which every equation of the 3rd degree is reducible, is (231. (A))

(A) $$x^3 + p\,x + q = 0.$$

The analytical solution of this equation is

(E) $$x = \sqrt[3]{\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} + \sqrt[3]{\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)}$$

which may be reduced to the following form

$$x = \sqrt[3]{\left(-\frac{q}{2} + \frac{q}{2}\sqrt{1 + \frac{4\,p^3}{27\,q^2}}\right)} + \sqrt[3]{\left(-\frac{q}{2} - \frac{q}{2}\sqrt{1 + \frac{4\,p^3}{27\,q^2}}\right)},$$

substituting for $\frac{4\,p^3}{27\,q^2}$, $tan.^2\,B$, or

(F) . . . . . . $$tan.\,B = \frac{p}{3q} \times 2\sqrt{\frac{p}{3}}$$

there results

$$\sqrt[3]{\left[-\frac{q}{2}\left(1 - \frac{1}{cos.\,B}\right)\right]} + \sqrt[3]{\left[-\frac{q}{2}\left(1 + \frac{1}{cos.\,B}\right)\right]}$$

CCXXXVII. For greater conciseness, let $2\sqrt{\frac{p}{3}} = R$, which gives $p = \frac{3}{4}R^2$, the equation (F) then becomes, $tan.\,B = \frac{R^3}{4q}$; we therefore obtain $q = \frac{R^3}{4\,tan.\,B}$. Substituting this value of q; we have

$$x = \sqrt[3]{-\frac{R^3}{8\,tan.B} \times \frac{cos.B - 1}{cos.\,B}} + \sqrt[3]{-\frac{R^3}{8tan.\,B} \times \frac{cos.B + 1}{cos.\,B}} =$$

$$\frac{R}{2}\left(\sqrt[3]{\frac{1 - cos.B}{sin.B}} - \sqrt[3]{\frac{1 + cos\,B}{sin.\,B}}\right) = \frac{R}{2}\left(\sqrt[3]{tan.\frac{B}{2}} - \sqrt[3]{cot.\frac{B}{2}}\right) =$$

or $x = -R \times \dfrac{\sqrt[3]{cot. \frac{B}{2}} - \sqrt[3]{tan. \frac{B}{2}}}{2}$; but, by a well known formula, $cot. 2A = \dfrac{cot. A - tan. A}{2}$, comparing these two last equations, and making

$$(G) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad tan. A = \sqrt[3]{tan. \frac{B}{2}}$$

and consequently $cot. A = \sqrt[3]{cot. \frac{B}{2}}$, we have $x = - R. \times cot. 2A$, and replacing the value of R, we get the final equation (K) $\quad . \quad . \quad . \quad . \quad x = - 2\sqrt{\frac{p}{3}}. cot. 2 A.$

Thus we obtain the value of x by means of three very simple equations: 1st, by equation (F) we find the arc B; 2nd, by equation (G), we get the arc A, and 3rd, by equation (K) we obtain the value of x.

CCXXXVIII. If the proposed equation were of the form

$$x^3 + p x - q = 0,$$

then the analytical solution (E) $-\frac{q}{2}$ would become positive. Pursuing the same method as above, we should obtain the same equations (F), (G), (K), with the simple difference, that the second member of the last equation would have a positive sign.

CCXXXIX. Again, let the proposed equation be of the form

$$x^3 - p x + q = 0,$$

which renders $\frac{p^3}{27}$ negative in the analytical solution (E); we then have

$$x = \sqrt[3]{\left(-\frac{q}{2} + \frac{q}{2}\sqrt{1 - \frac{4p^3}{27q^2}}\right)} + \sqrt[3]{\left(-\frac{q}{2} - \frac{q}{2}\sqrt{1 - \frac{4p^3}{27q^2}}\right)}.$$

Making $\frac{4\,p^3}{27\,q^2} = sin.^2\, B$; or

$$(L) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad sin.\ B = \frac{p}{3\,q} \times 2\sqrt{\frac{p}{3}};$$

there results $x = \sqrt[3]{\left[-\frac{q}{2},(1-cos. B)\right]} + \sqrt[3]{\left[-\frac{q}{2},(1+cos. B)\right]}$.

But the equation (L) gives $q = \frac{R^3}{4\,sin.\ B}$, and making as before

$R = 2\sqrt{\frac{p}{3}}$, and consequently $p = \frac{3}{4}R^2$, we have

$$x = \sqrt[3]{-\frac{1}{8}R.^3\frac{1-cos.\ B}{sin.\ B}} + \sqrt[3]{-\frac{1}{8}R^3,\frac{1+cos. B}{sin.\ B}} = -$$

$$R.\frac{\sqrt[3]{tan.\frac{B}{2}} + \sqrt[3]{cot.\ \frac{B}{2}}}{2}.$$ But $\frac{tan.\ A + cot.\ A}{2} = \frac{}{sin.\ 2\ A}$,

and employing equation (G), we obtain

$$x \quad \frac{R}{sin.\ 2\ A} \quad \text{or} \quad (M) \quad . \quad . \quad . \quad x = -\frac{2\sqrt{\frac{p}{3}}}{sin.\ 2\,A}.$$

We then find the value of x by means of the three equations (L), (G), (M).

CCXL. It is obvious that the above method will also serve, when the equation to be resolved is of the form

$$x^3 - p\,x - q = 0,$$

with this difference, that the 2nd member of the equation (M) then becomes positive.

In the two last equations, it is necessary that $\frac{4\,p^3}{27\,q^2}$ be a quantity not greater than 1, else we could not suppose $\frac{4p^3}{27q^2} = sin.^2 B$, as a sine can never be greater than the radius.

CCXLI. In the irreducible case, when p being negative, $4\,p^3 > 27\,q^2$ or $\frac{p^3}{27} > \frac{q^2}{4}$, the quantity $\sqrt{1 - \frac{4\,p^3}{27\,q^2}}$ presenting itself as remarked before, under an imaginary form whenever all the three roots are real, offers no difficulty; indeed, in trigonometry it is perfectly indifferent whether the roots be rational or irrational, both being found with an equal facility.

To obtain these, we have only to abandon the analytical solution, embarrassed with imaginary quantities, and to select from amongst the trigonometrical formulæ, such as are comparable to the equation

$$(N) \quad \ldots\ldots\ldots\ldots \quad x^3 - p\,x \pm q = 0.$$

CCXLII. Two formulæ are of that nature, viz. $sin.\,3\,A = 3\,sin.\,A - 4\,sin.^3\,A$, and $cos.\,3\,A = 4\,cos.^3\,A - 3\,cos.\,A$, making the first homogeneous* and dividing by 4, we have

$$(O) \quad \ldots \quad sin^3.\,A - \frac{3}{4}R^2\,sin.\,A + \frac{1}{4}\,R^2\,sin.\,3\,A = 0.$$

Comparing this equation, term for term, with the one proposed, (N) assuming first q positive; making $x = sin.\,A$, we have $1°$, $p = \frac{3}{4}\,R^2$, which gives $R^2 = \frac{4}{3}\,p$ and $R = 2\sqrt{\frac{p}{3}}$; $2°$ $q = \frac{1}{4}\,R^2\,sin.\,3\,A = \frac{p}{3}\,sin.\,3\,A$; whence we get $sin.\,3\,A = \frac{3\,q}{p}$. This sine being proportional to the radius of the equation, which is, as previously stated, $2\sqrt{\frac{p}{3}}$, we must then, in order

* An *homogeneous equation* or an *equation of equal dimensions*, is that which contains in each term an equal number of algebraic or geometric factors (without having however regard to the numerical coefficients). If R were preserved in all the trigonometrical operations, we should find in each formula every term having an equal dimension; but it is easy to make them such, by restoring to each the factor R raised to the power necessary to make them homogeneous. For example, calling x and y two trigonometrical lines, an equation of the form of $4\,x^3 = 3x - y + 2$; x^3 is of three dimensions, x and y of one and 2 without any dimension; in order to introduce the radius, regularly in this equation, we must write, $4\,x^3 = 3\,R^2\,x - R^2\,y + 2\,R^3$; and such would be the expression of this equation, if the R's had not been made to disappear by substituting units for them.

to find it in the tables, divide it by this same radius, thus we have

$$(P) \quad . \quad . \quad . \quad sin.\,3\,A = \frac{3\,q}{p}.\frac{1}{2\sqrt{\frac{p}{3}}}.$$

CCXLIII. The value of $3A$ being now known, that of A is also known, and consequently that of $x = sin.\ A$. But as $sin.\ A$ is taken from tables where $R = 1$, we must afterwards multiply it by the radius of the equation. Consequently

$$(Q) \qquad x = sin.\,A \times 2\sqrt{\frac{p}{3}}.$$

CCXLIV. If in the equation (O), we substitute $360 + 3\,A$ for $3\,A$, we must also substitute $120 + A$ for A. But $sin.\ (360° + 3\,A) = sin.\ 360°\ cos.\ 3\,A + cos.\ 360\ sin.\ 3\,A = sin.\ 3\,A$; the equation ($P$) is then the same when instead of $x = sin.\ A$, we suppose $x = sin.\ (120 + A)$. This then is a second value of x which satisfies the conditions of the equation (O). But $sin.\ (120 + A) = sin.\ (60° - A)$. Hence we have

$$(S) \qquad sin.\,(60° - A) \times 2\sqrt{\frac{p}{3}}.$$

CCXLV. Again, if in the equation (O), we substitute $720° + 3\,A$ for $3\,A$, which gives $240° + A$ for A, we have also $sin.\ (720 + 3\,A) = sin.\ [360 + (360 + 3\,A)] = sin.\ 3\,A$; and the equation ($P$) subsisting still, we have a third value for x, which is

$$(T) \qquad - sin.\,(60° + A)\,2\sqrt{\frac{p}{3}}.$$

The equation (P) evidently supposes that we can not have $4\,p^3 \angle\ 27\,q^2$. These equations Q, S and T, are therefore only applicable to the cases resolved in art. (238, 239), when $4\,p^3 = 27q^2$ exactly, then the formulæ (Q) and (S) give two equal values of x, and the formula (T) a third value, whilst the method furnished in art. (238), gives only one value of x, viz. the last (T).

CCXLVI. The great advantage of trigonometrical solutions is at once apparent, when we consider that, by means of the four equations (P) (Q) (S) (T) which have the common factor

$2\sqrt{\frac{p}{3}}$, we readily obtain the three values of x, either exact or approximate, values which by analysis can only be obtained through a very laborious process.

These same four equations, by changing the sign of the second member of the last three, give also the values sought, when q is negative, and the equation to be resolved is

$$x^3 - p\,x - q = 0,$$

in this case $sin.3A$ becomes negative, we then substitute $180° + 3\,A$ for $3\,A$, and consequently $60° + A$, for A. The formulæ then become reversed, with merely a change in the sign, viz. Q in T, S in Q and T in S.

CCXLVII. It may here be remarked, that in adding more than twice the circumference of the circle (244, 245), we could never get other values of x than the three already found; which confirms the theory of equations, according to which the equation (O) can have no more than three roots.

There are consequently three sines which resolve the equation (O), viz. $sin.\ A$, $sin.\ (60° - A)$ and $-\ sin.(60° + A)$.

CCXLVIII. *Examples of the resolution of equations of the 3rd degree, by means of trigonometry.*

Let the given equation, be

$$x^3 + 2\,x + 33 = 0;$$

this case comes under formula (A), here we have

$$tan.\ B = \frac{2}{3 \times 33} \times 2\sqrt{\frac{2}{3}} = \frac{2}{.99} \times \sqrt{\frac{8}{3}}$$

$$tan.\ A = \sqrt[3]{tan.\frac{B}{2}}$$

$$x = -\sqrt{\frac{8}{3}}.\ cot.2A$$

Calculation.

$log.\ 8$	. . . . . =	0,9030900
$log.\ 3$	. . . . . =	4,4771213
$log.\ \frac{8}{3}$	. . . . . =	0,4259687
$log.\ \sqrt{\frac{8}{3}}$	. . . . =	0,2129844

$$log. \sqrt{\frac{8}{3}} \quad . \quad . \quad = 0,2129844$$

$$log.\ 2 \quad . \quad . \quad . \quad . \quad = 0,3010300$$

$$compl.\ log.\ 99 \quad . \quad . \quad . \quad = 8,0043648$$

$$log.\ tan.\ B \quad . \quad . \quad = 8,5183792 \therefore B = 1^\circ\ 53'\ 22''\ 2$$

$$log.\ tan.\ \frac{B}{2} \quad . \quad . \quad = 8,2172311$$

$$\frac{log.\ tan.\ \frac{B}{2}}{3} = log.\ tan.\ A \quad = 9,4057437 \therefore A = 14^\circ\ 16'\ 49''\ 8$$

$$log.\ cot.\ 2A \quad . \quad . \quad = 0,2641368$$

$$log. \sqrt{\frac{8}{3}} \quad . \quad . \quad = 0,2129844$$

$$* \therefore log.\ (-x) \quad . \quad . \quad = 0,4771212 = log. - 3.$$

CCXLIX. Let it be required to resolve the equation

$$x^3 - \frac{403}{441}x + \frac{46}{147} = 0;$$

in this case, p being minus, we must see first how many real roots the equation has. Now $4p^3 = 4 \times \left(\frac{403}{441}\right)^3$, and $27q^2 = 27 \times \left(\frac{46}{147}\right)^2$; then $log.\ 4\ p^3 = 0,485$, $log.\ 27q^2 = 0,422$. Therefore $4\ p^3 > 27\ q^2$. This then is the case which to analysis is *irreducible*.

By equation (P) we have

$$sin.\ 3\ A = \frac{3 \times 46}{147} \times \frac{451}{403} \times \frac{1}{2\sqrt{\frac{403}{3 \times 441}}} = \frac{414}{403} \times \frac{1}{\sqrt{\frac{1612}{1323}}}$$

The three values of x are consequently

$$1\text{st } x = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad sin.\ A \sqrt{\frac{1612}{1323}}$$

$$2\text{nd } x = \quad . \quad . \quad . \quad sin.\ (60^\circ - A) \sqrt{\frac{1612}{1323}}$$

$$3\text{rd } x = \quad . \quad . \quad - sin.\ (60^\circ + A) \sqrt{\frac{1612}{1323}}$$

* The expression $log.\ (-x)$ signifies simply that the negative sign is to be prefixed to the number corresponding.

Calculation.

$$
\begin{array}{lll}
log.\ 1612 \ldots & = 3,2073650 & \\
log.\ 1323 \ldots & = 3,1215598 & \\
\hline
log. \dfrac{1612}{1323} \ldots\ldots & = 0,0858052 & \\
log. \sqrt{\dfrac{1612}{1323}} & = 0,0429026 & = constant\ log \\
\hline
compl.\ of\ constant\ log. & = 9.9570974 & \\
log.\ 414 \ldots & = 2,6170003 & \\
compl.\ log.\ 403 \ldots & = 7,3946950 & \\
\hline
log.\ sin.\ 3\ A \ldots & = 9,9687927 & = log,\ sin.\ 68^{\circ}\ 32''\ 18'6 \\
\hline
log.\ sin.\ A\ (22^{\circ}\ 50'\ 46''\ 2) & = 9,5891206 & \\
log.\ constant \ldots & = 0,0429026 & \\
\hline
1st\quad log.\ x \ldots & = 9,6320232 & = 0,4285714 \\
\hline
log.\ sin.\ (60^{\circ} - A) & = 9,7810061 & \\
log.\ constant \ldots & = 0,0429026 & \\
\hline
2nd\quad log.\ x \ldots & = 9,8239087 & = 0,6666666 \\
\hline
log.\ sin.\ (60^{\circ} + A) \ldots & = 9,9966060 & \\
log.\ constant \ldots & = 0,0429026 & \\
\hline
3rd\quad log. - x \ldots & = 0,0395086 & = -1,095238.
\end{array}
$$

CCL. It will be observed in the first place that the negative value of x is the sum of the two positive as it ought to be, which therefore proves the exactness of the calculation. Again, the second value of x is visibly $\frac{2}{3}$.

To know whether the other two values of x can be expressed exactly by vulgar fractions, we have recourse to the same method as in art. (128), consequently for the 1st value of x, $compl.\ log.\ x = 0,3679768 = \frac{1}{2,33333}$, it will be readily seen that by multiplying this last fraction by 3, we get $x = \frac{3}{7}$. Indeed, the $log.$ of $\frac{3}{7}$ is 9,6320232 which is precisely the $log.$ of the first value of x. But the negative value of x ought to be equal to the sum of the two positive values taken with contrary sign,

we must have therefore, $-\left(\frac{2}{3}+\frac{3}{7}\right)=-\frac{23}{21}$. We have thus for the three values of x in the proposed equation $=\frac{2}{3}$; $+\frac{3}{7}$ and $-\frac{23}{21}$. Substituting these separately, we shall be convinced of their correctness.

CCLI. Before closing the subject we may here recapitulate the *solutions of all the equations of the 2nd* and *3rd degree by means of trigonometry.*

Equations of the 2nd Degree.

1st case x^2+px+q. If $p^2 \angle 4q$, x is imaginary. $Sin.\,A=\frac{2}{p}\sqrt{q}$

$$x=-\sqrt{q}.\,tan.\,\frac{A}{2};\quad x=-\sqrt{q}.\,cot.\,\frac{A}{2}$$

2nd case x^2+px-q . . . . $tan.\,A=\frac{2}{p}\sqrt{q}$

$$x=\sqrt{q}.\,tan.\,\frac{A}{2};\quad x=-\sqrt{q}.\,cot.\,\frac{A}{2}$$

3rd case x^2-px+q. If $p^2 \angle 4q$, x is imaginary. $Sin.\,A=\frac{2}{p}\sqrt{q}$

$$x=\sqrt{q}.\,tan.\,\frac{A}{2};\quad x=\sqrt{q}.\,cot.\,\frac{A}{2}$$

4th case x^2-px-q . . . . $tan.\,A=\frac{2}{p}\sqrt{q}$

$$x=-\sqrt{q}.\,tan.\,\frac{A}{2};\quad x=\sqrt{q}.\,cot.\,\frac{A}{2}.$$

Equations of the 3rd Degree.

1st and 2nd cases $x^3+px\pm q=0$ $\begin{cases} tan.\,B=\frac{p}{3q},\ 2\sqrt{\frac{p}{3}} \\ tan.\,A=\sqrt[3]{tan.\,\frac{B}{2}} \end{cases}$

One real root . . $x=\mp 2\sqrt{\frac{p}{3}}\,cot.\,2A$

3rd and 4th cases $x^3 - p\,x \pm x = 0$ supposing $4\,p^3 \angle 27\,q^2$ $\left\{\begin{array}{l} sin.\ B = \frac{p}{3\,q}.\ 2\sqrt{\frac{p}{3}} \\ tan.\ A = \sqrt[3]{tan.\ \frac{B}{2}} \end{array}\right.$

One real root . . $x = \mp \dfrac{2\sqrt{\frac{p}{3}}}{sin.2A}$

The two cases, called irreducible :

$$4\,p^3 > \text{or} = 27\,q^2.$$

1st case, $x^3 - p\,x + q = 0$. $sin.\ 3\,A = \frac{3\,q}{p}.\ \frac{1}{2\sqrt{\frac{p}{3}}}$

$x = 2\sqrt{\frac{p}{3}}\ sin.\ A, x = 2\sqrt{\frac{p}{3}}\ \ sin.\ (60^\circ - A), x = -2\sqrt{\frac{p}{3}}.$ $sin.\ (60^\circ + A)$

2nd case, $x^3 - p\,x - q = 0$ . . $sin.\ 3\,A = \frac{3\,q}{p}.\ \frac{1}{2\sqrt{\frac{p}{3}}}.$

$x = -2\sqrt{\frac{p}{3}}\ sin.\ A, x = -2\sqrt{\frac{p}{3}}.\ sin.\ (60^\circ - A),\ x = 2\sqrt{\frac{p}{3}}\ sin.\ (60^\circ + A).$

Of Proportion and Progression.

CCLII. Arithmetic introduces us to a knowledge of the definition and fundamental properties of *proportion* and *equidifference*, or of what is termed *geometrical and arithmetical proportion*. We now proceed to treat of the application of algebra to the principles there developed; this will lead to several results, of which frequent use is made in geometry.

We shall begin by observing, that equidifference and proportion may be expressed by equations. Let A, B, C, D, be the four terms of the former, and a, b, c, d, the four terms of the latter; we have then

$$B - A = D - C;\ \text{and}\ \frac{b}{a} = \frac{d}{c},$$

equations, which are to be regarded as equivalent to the expressions

$$A\,.\,B : C\,.\,D,\quad a : b :: c : d,$$

and which give

$$A + D = B + C, \quad a\,d = b\,c.$$

Hence it follows, that, *in equidifference, the sum of the extremes terms is equal to that of the means, and in proportion, the product of the extremes is equal to the product of the means*, as is shown in works on arithmetic by reasonings, of which the above equations are only a translation into algebraic expressions.

The reciprocal of each of the preceding propositions may be easily demonstrated; for from the equations

$$A + D = B + C, \quad a\,d = b\,c,$$

we return at once to

$$D - C = B - A, \quad \frac{b}{a} = \frac{d}{c},$$

and, consequently, *when four quantities are such, that two among them give the same sum, or the same product, as the other two, the first are the means and the second the extremes* (or the converse) *of an equidifference or proportion.*

When $B = C$, the equidifference is said to be *continued;* the same is said of proportion, when $b = c$. We have in this case

$$A + D = 2\,B, \quad a\,d = b^2;$$

that is, *in continued equidifference, the sum of the extremes is equal to double the mean;* and *in proportion, the product of the extremes is equal to the square of the mean.* From this we deduce

$$B = \frac{A + D}{2}, \quad b = \sqrt{a\,d};$$

the quantity B is the *middle* or mean arithmetical proportional between A and D, and the quantity b the *mean geometrical proportional* between a and d.

The fundamental equations,

$$B - A = D - C, \quad \frac{b}{a} = \frac{d}{c},$$

lead also to the following;

$$C - A = D - B, \quad \frac{c}{a} = \frac{d}{b};$$

from which it is evident, that we may change the relative places of the means in the expressions $A \,.\, B : C \,.\, D$, $a : b :: c : d$, and in this way obtain $A \,.\, C : B \,.\, D$, $a : c :: c : d$. In general, we may make any transposition of the terms which is consistent with the equations

$$A + D = B + C \text{ and } a\,d = b\,c.$$

We have now done with equidifference, and shall proceed to consider proportion simply.

CCLIII. It is evident, that to the two members of the equation $\frac{b}{a} = \frac{d}{c}$ we may add the same quantity m, or subtract it from them; so that we have

$$\frac{b}{a} \pm m = \frac{d}{c} \pm m;$$

reducing the terms of each member to the same denominator, we obtain

$$\frac{b \pm m\,a}{a} = \frac{d \pm m\,c}{c},$$

an equation, which may assume the form

$$\frac{c}{a} = \frac{d \pm m\,c}{b \pm m\,a},$$

and may be reduced to the following proportion,

$$b \pm ma : d \pm mc :: a : c;$$

and as $\frac{c}{a} = \frac{d}{b}$, we have likewise

$$\frac{d \pm m\,c}{b \pm m\,a} = \frac{d}{b},$$

or $$b \pm ma : d \pm mc :: b : d.$$

These two proportions may be enunciated thus; *The first consequent plus or minus its antecedent taken a given number of times, is to the second consequent plus or minus its antecedent taken the same number of times, as the first term is to the third, or as the second is to the fourth.*

Taking the sums separately and comparing them together, and also the differences, we obtain

$$\frac{d + m\,c}{b + m\,a} = \frac{c}{a}, \quad \frac{d - m\,c}{b - m\,a} = \frac{c}{a},$$

whence we conclude

$$\frac{d + m\,c}{b + m\,a} = \frac{d - m\,c}{b - m\,a},$$

that is,

$$b + m\,a : d + m\,c :: b - m\,a : d - m\,c;$$

or rather, by changing the relative places of the means

$$b + m\,a : b - m\,a :: d + m\,c : d - m\,c;$$

and if we make $m = 1$, we have simply

$$b + a : b - a :: d + c : d - c,$$

which may be enunciated thus;

The sum of the first two terms, is to their difference, as the sum of the last two, is to their difference.

CCLIV. The proportion $a : b :: c : d$ may be written thus:

$$a : c :: b : d;$$

we have then

$$\frac{c}{a} \pm m = \frac{d}{b} \pm m,$$

whence

$$\frac{c \pm m\,a}{a} = \frac{d \pm m\,b}{b},$$

and lastly,

$$c \pm m\,a : d \pm m\,b :: a : b \text{ or } :: c : d,$$

from which it follows, that *the second antecedent plus or minus the first taken a given number of times, is to the second consequent plus or minus the first taken the same number of times, as any one of the antecedents whatever, is to its consequent.*

This proposition may also be deduced immediately from that given in the preceding article; for by changing the order of the means in the original proportion

$$a : b :: c : d,$$

and applying the proposition referred to, we obtain, successively,

$$a : c :: b : d$$

$$c \pm m\,a : d \pm m\,b :: a : b \text{ or } :: c : d,$$

and denominating the letters, a, b, c, d, in this last proportion, according to the place they occupy in the original proportion, we may adopt the preceding enunciation.

Making $m = 1$, we obtain the proportions

$$c \pm a : d \pm b :: a : b$$
$$:: c : d$$
$$c + a : c - a :: d + b : d - b;$$

whence it appears, that *the sum or difference of the antecedents, is to the sum or difference of the consequents, as one antecedent, is to its consequent*; and that *the sum of the antecedents, is to their difference, as that of the consequents, is to their difference.*

In general, if we have

$$\frac{b}{a} = \frac{d}{c} = \frac{f}{e} = \frac{h}{g}, \text{ \&c.}$$

and make $\frac{b}{a} = q$, we shall have

$$\frac{d}{c} = q, \frac{f}{e} = q, \frac{h}{g} = q, \text{ \&c.}$$

which gives

$$b = a\,q, \; d = c\,q, \; f = e\,q, \; h = g\,q, \text{ \&c.}$$

then by adding these equations, member to member, we obtain

$$b + d + f + h = a\,q + c\,q + e\,q + g\,q,$$

or

$$b + d + f + h = q\,(a + c + e + g).$$

whence it follows, that

$$\frac{b + d + f + h}{a + c + e + g} = q = \frac{b}{a}.$$

This result is enunciated thus; *in a series of equal ratios,*

a : b : : c : d : : e : f : : g : h, &c.

the sum of any number whatever of antecedents, is to the sum of a like number of consequents, as one antecedent, is to its consequent.

CCLV. If we have the two equations

$$\frac{b}{a} = \frac{d}{c}, \text{ and } \frac{f}{e} = \frac{h}{g},$$

and multiply the first members together and the second together, the result will be

$$\frac{bf}{ae} = \frac{dh}{cg};$$

an equation equivalent to the proportion

$$ae : bf :: cg : dh,$$

which may be obtained also by multiplying the several terms of the proportion

$$a : b :: c : d,$$

by the corresponding ones in the proportion

$$e : f :: g : h.$$

Two proportions multiplied thus, term by term, are said to be *multiplied in order*; and the products obtained in this way, are, as will be seen, proportional; the new ratios are the ratios *compounded* of the original ratios.

It will be readily perceived also, that if we divide two proportions term by term, or in *order*, the result will be a proportion.

CCLVI. If we have

$$\frac{b}{a} = \frac{d}{c}, \text{ or } \frac{b\,b\,b}{a\,a\,a} = \frac{d\,d\,d}{c\,c\,c}$$

we may deduce from it

$$\frac{b^3}{a^3} = \frac{d^3}{c^3} \text{ or } \frac{b^m}{a^m} = \frac{d^m}{c^m},$$

which gives

$$a^m : b^m :: c^m : d^m;$$

whence it follows, that *the squares, the cubes, and, in general, the similar powers of four proportional quantities are also proportional.*

The same may be said of fractional powers, for, since

$$\sqrt[m]{\frac{b}{a}} = \frac{\sqrt[m]{b}}{\sqrt[m]{a}}, = \sqrt[m]{\frac{d}{c}}$$

and

$$\sqrt[m]{\frac{d}{c}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}};$$

therefore,

$$\frac{\sqrt[m]{b}}{\sqrt[m]{a}} = \frac{\sqrt[m]{d}}{\sqrt[m]{c}},$$

or

$$\sqrt[m]{a} \quad \sqrt[m]{b} :: \sqrt[m]{c} : \sqrt[m]{d},$$

if $a : b :: c : d$; that is, *the roots of the same degree of four proportional quantities, are also proportional.*

Such are the leading principles in the theory of proportion. This theory was invented for the purpose of discovering certain quantities by comparing them with others. Latin names were for a long time used to express the different changes or transformations, which a proportion admits of. We are beginning to relieve the memory of the mathematical student from so unnecessary a burden; and this parade of proportions might be entirely superseded by substituting the corresponding equations, which would give greater uniformity to our methods, and more precision to our ideas.

CCLVII. We pass from proportion to progression by an easy transition. After we have acquired the notion of three quantities in continued equidifference, the last of which exceeds the first, we shall be able, without difficulty, to represent to ourselves an indefinite number of quantities, a, b, c, d, &c. such, that each shall exceed the preceding one, by the same quantity δ, so that

$$b = a + \delta,\ c = b + \delta,\ d = c + \delta,\ e = d + \delta,\ \&c.$$

A series of these quantities is written thus;

$$\div\ a \,.\, b \,.\, c \,.\, d \,.\, e \,.\, f,\ \&c.$$

and is termed an *arithmetical progression*; we have thought it proper, however, to change this denomination to that of *progression by differences.*

We may determine any term whatever of this progression, without employing the intermediate ones. In fact, if we substitute for b its value, in the expression for c, we have

$$c = a + 2\,\delta;$$

by means of this last, we find

$$d = a + 3\,\delta, \text{ then } e = a + 4\,\delta,$$

and so on; whence it is evident, that representing by z the last term, the place of which is denoted by n, we have

$$z = a + (n - 1)\,\delta.$$

Let there be for example, the progression

$$\div\ 3 \,.\, 5 \,.\, 7 \,.\, 9 \,.\, 11 \,.\, 13 \,.\, 15 \,.\, 17,\ \&c.$$

here the first term $a = 3$, the difference or *ratio* $\delta = 2$; we find for the eighth term,

$$3 + (8 - 1)\,2 = 17,$$

the same result, to which we arrive by calculating the several preceding terms.

The progression we have been considering is called *increasing*; by reversing the order, in which the terms are written thus,

$$\div\ 17 \,.\, 15 \,.\, 13 \,.\, 11 \,.\, 9 \,.\, 7 \,.\, 5 \,.\, 3 \,.\, 1 \,.\, -1 \,.\, -3, \text{ \&c.}$$

we form a *decreasing* progression. We may still find any term whatever by means of the formula $a + (n - 1)\,\delta$, observing only, that δ is to be considered as negative, since, in this case, we must subtract the difference from any particular term in order to obtain the following.

CCLVIII. We may also, by a very simple process, determine the sum of any number whatever of terms in a progression by differences. This progression being represented

$$\div\ a \,.\, b \,.\, c \,.\, \ldots\ldots\ldots \,.\, x \,.\, y \,.\, z,$$

and s denoting the sum of all the terms, we have

$$s = a + b + c \ldots\ldots\ldots + x + y + z.$$

Reversing the order, in which the terms of the second member of this equation are written, we have still

$$s = z + y + x \ldots\ldots\ldots\ldots + c + b + a.$$

If we add together these equations, and unite the corresponding terms, we obtain

$$2s = (a+z)+(b+y)+(c+x)\ldots+(x+c)+(y+b)+(z+a);$$

but by the nature of the progression, we have, beginning with the first term,

$$a + \delta = b,\ b + \delta = c,\ \ldots\ldots\ x + \delta = y,\ y + \delta = z,$$

and, consequently, beginning with the last

$$z - \delta = y,\ y - \delta = x,\ \ldots\ldots\ c - \delta = b,\ b - \delta = a;$$

by adding the corresponding equations, we shall perceive at once, that

$$a + z = b + y = c + x, \text{ \&c.}$$

and, consequently, that

$$2\,s = n\,(a + z);$$

whence it follows, that

$$s = \frac{n\,(a + z)}{2}.$$

Applying this formula to the progression

$$\div\ 3\ .\ 5\ .\ 7\ .\ 9,\ \&c.$$

we find for the sum of the first eight terms

$$\frac{(3 + 17)\,8}{2} = 80.$$

CCLIX. The equation

$$z = a + (n - 1)\delta,$$

together with

$$s = \frac{(a + z)\,n}{2},$$

furnishes us with the means of finding any two of the five quantities, a, δ, n, z, and s, when the other three are known; we shall not stop to treat of the several cases, which may be presented.

CCLX. From proportion is derived progression by *quotients* or *geometrical* progression, which consists of a series of terms, such, that the quotient arising from the division of one term by that which precedes it, is the same, from whatever part of the series the two terms are taken. The series

$$\div\!\!\div\ 2 : 6 : 18 : 54 : 162\quad \&c.$$
$$\div\!\!\div\ 45 : 15 : 5 : \tfrac{5}{3} : \tfrac{5}{9}\quad \&c.$$

are progressions of this kind; the quotient or *ratio* is 3 in the first, and $\frac{1}{3}$ in the other; the first is increasing, and the second decreasing. Each of these progressions forms a series of equal ratios, and for this reason is written, as above.

Let

$$a,\ b,\ c,\ d,\ \ldots\ldots\ldots\ y,\ z,$$

be the terms of a progression by quotients: making $\frac{b}{a} = q$, we have by the nature of the progression,

$$q = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d}\ \ldots = \frac{z}{y},$$

or $b = a\,q$, $c = b\,q$, $d = c\,q$, $e = d\,q$, $\ldots$ $z = y\,q$.

Substituting, successively, the value of b in the expression for c, and the value of c in the expression for d, &c. we have

$$b = a\,q,\ c = a\,q^2,\ d = a\,q^3,\ e = a\,q^4. \quad . \quad . \quad z = a\,q^{n-1},$$

taking n to represent the place of the term l, or the number of terms considered in the proposed progression; the series is therefore converted into the following,

$$a + a\,q + a\,q^2 + a\,q^3 + a\,q^4, \quad . \quad . \quad . \quad a\,q^{n-1}.$$

By means of the formula $z = a\,q^{n-1}$ we may determine any term whatever, without making use of the several intermediate ones. The tenth term of the progression

$$\div\, 2 : 6 : 18 : \&c.$$

for example, is equal to $2 \times 3^9 = 39366$.

CCLXI. We may also find the sum of any number of terms we please of the progression

$$\div\, a : b : c : d, \&c.$$

by adding together the equations

$$b = a\,q,\ c = b\,q,\ d = c\,q,\ e = d\,q, \quad . \quad . \quad . \quad z = y\,q;$$

for the result will be

$$b + c + d + e \ . \ . + z = (a + b + c + d \ . \ . + y)\,q;$$

and representing by s the sum sought, we have

$$b + c + d + e \quad . \quad . \quad . \quad + z = s - a,$$
$$a + b + c + d \quad . \quad . \quad . \quad + y = s - z,$$

whence $$s - a = q\,(s - z),$$

and, consequently,

$$s = \frac{q\,z - a}{q - 1}.$$

In the above example, we find for the sum of the first ten terms of the progression

$$\div\, 2 : 6 : 18 : \&c.$$

$$\frac{2 \times 3^{10} - 2}{2} = 3^{10} - 1 = 59048.$$

CCLXII. The two equations

$$z = a\,q^{n-1},\ s = \frac{q\,z - a}{q - 1},$$

comprehend the mutual relations, which exist among the five quantities, a, q, n, z, and s, in a progression by quotients, and enable us to find any two of these quantities, when the other three are given.

CCLXIII. If we substitute $a\,q^{n-1}$ in the place of z, in the expression for s, we have

$$s = \frac{a\,(q^n - 1)}{q - 1}.$$

When q surpasses unity, the quantity q^n will become greater and greater in proportion to the increased magnitude of the number n; and s may be made to exceed any quantity whatever, by assigning a proper value to n, that is, by taking a sufficient number of terms in the proposed progression. But if q is a fraction, represented by $\frac{1}{m}$, we have

$$s = \frac{a\left(\frac{1}{m^n} - 1\right)}{\frac{1}{m} - 1} = \frac{a\,m\left(1 - \frac{1}{m^n}\right)}{m - 1} = \frac{a\,m - \frac{a}{m^{n-1}}}{m - 1};$$

and it is evident, that as the number n becomes greater, the term $\frac{a}{m^{n-1}}$ will become smaller, and consequently, the value of s will approach nearer and nearer to the quantity $\frac{a\,m}{m-1}$ from which it will differ only by

$$\frac{a}{(m-1)\,m^{n-1}};$$

therefore, the greater the number of terms we take in the proposed progression, the more nearly will their sum approach to $\frac{a\,m}{m-1}$. It may even differ from $\frac{a\,m}{m-1}$ by a quantity less than any assignable quantity, without ever becoming, in a rigorous sense, equal to it.

The quantity $\frac{a\,m}{m-1}$, which we shall designate by l, forms, we perceive, a limit, to which the particular sums represented by s, approach nearer and nearer.

Applying what has been said to the progression

$$\div 1 : \tfrac{1}{2} : \tfrac{1}{4} : \tfrac{1}{8} : \tfrac{1}{16}, \&c.$$

we have

$$a = 1,\ q = \frac{1}{m} = \tfrac{1}{2},$$

whence

$$m = 2,\ l = \frac{a\,m}{m-1} = 2;$$

and the greater number of terms we take in the above progression, the nearer their sum will approach to an equality with 2.

We have, in fact,

$$\begin{aligned}
&1 &&= 1 = 2 - 1,\\
&1 + \tfrac{1}{2} &&= \tfrac{3}{2} = 2 - \tfrac{1}{2},\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} &&= \tfrac{7}{4} = 2 - \tfrac{1}{4},\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= \tfrac{15}{8} = 2 - \tfrac{1}{8},\\
&1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} &&= \tfrac{31}{16} = 2 - \tfrac{1}{16}.
\end{aligned}$$

&c.

The expression for l may be considered as the sum of the decreasing progression by quotients, continued to infinity, and it is thus, it is usually presented; but in order to form a clear idea of it, we must represent it in a limited view.

The same result will be obtained, by the following process.

Let it be proposed to sum up the decreasing geometrical series

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ . . . to an infinite number of terms, here the ratio $= \frac{1}{2}$; multiplying each term of this series by the reciprocal of the ratio, we make each term equal to its preceding, and will form a second series; subtracting then the first from the second, we get the difference of the two series, from which we easily obtain the sum of the first series; let s represent the sum, then

$$s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \&c.$$

multiplying both numbers of this equation, by the reciprocal of the ratio $\frac{1}{2}$, that is by 2, we have,

$$2\,s = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$$

as each series never terminates, but is continued *ad infinitum*, all the terms of the second series, with the exception of the first, are cancelled by all the terms of the first series, we have therefore $2s - s = 2$ or $s = 2$.

Be the ratio of the following series $\frac{1}{3}$, we have

$$s = 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \&c.$$

$$3s = 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \&c.$$

$$\therefore 3s - s = 9 \qquad \therefore \qquad s = \frac{9}{2} = 4\frac{1}{2}.$$

Again let the common ratio be $\frac{3}{7}$

$$s = 6 + \frac{18}{7} + \frac{54}{49} + \frac{162}{343} + \&c.$$

$$\frac{7}{3}s = 14 + 6 + \frac{18}{7} + \frac{54}{49} + \&c.$$

$$\frac{7}{3}s - s = 14 \qquad \therefore \qquad s = 10\frac{1}{2}.$$

CCLXIV. We may obtain from the expression (263).

$$s = \frac{a(q^n - 1)}{q - 1},$$

all the terms of the progression, of which it denotes the sum; for, if we divide $q^n -$ by $q - 1$ (166), we find

$$\frac{q^n - 1}{q - 1} = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + q^3 + q^4 \quad . \quad . \quad . \quad + q^{n-1},$$

which gives

$$s = a + aq + aq^2 \quad . \quad . \quad . \quad . \quad + aq^{n-1}.$$

We may employ the value of l for the same purpose; in this case, m is to be divided by $m - 1$, as follows:

$$
\begin{array}{l|l}
m & m-1 \\
\cline{2-2}
-m+1 & 1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c. \\
\cline{2-2}
\quad -1+\frac{1}{m} & \\
\qquad\quad -\frac{1}{m}+\frac{1}{m^2} & \\
\qquad\qquad\qquad -\frac{1}{m^2}+\frac{1}{m^3} & \\
\qquad\qquad\qquad \&c. &
\end{array}
$$

We begin, by dividing, according to the usual method, by the first term, and find 1 for the quotient; we multiply this quotient by the divisor and subtract the product from the dividend; then, dividing the remainder by the first term of the divisor, we obtain $\frac{1}{m}$ for the quotient, and have $\frac{1}{m}$ for a remainder; we go through the same process with this remainder as with the preceding. Pursuing this method, we soon discover the law, to which the several particular quotients are subjected, and perceive that the expression $\frac{m}{m-1}$, is equivalent to the series

$$1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c.$$

continued to infinity. Substituting for m its value $\frac{1}{q}$, and multiplying by a, we find as before

$$a+a\,q+a\,q^2+a\,q^3+\&c.$$

for the progression of which l represents the limit.

CCLXV. The series

$$1+\frac{1}{m}+\frac{1}{m^2}+\frac{1}{m^3}+\&c.$$

is considered as the value of the fraction $\frac{m}{m-1}$, whenever it is

converging, that is, when the terms, of which it is composed, become smaller and smaller the further they are removed from the first.

Indeed, if we make the division cease successively at the first, second, third, ... remainder, we have

the quotients	and the remainders
1	1
$1+\frac{1}{m}$	$\frac{1}{m}$
$1+\frac{1}{m}+\frac{1}{m^2}$	$\frac{1}{m^2}$
&c.	&c.

the former of which, approach the true value, exactly in proportion as the latter are diminished ; and this takes place, only when m exceeds unity. In all other cases we must have regard to the remainders, which, increasing without limit, make it evident, that the quotients are departing further and further from the true value.

To render this clear, we have only to make, successively, $m=2$, $m=1$, $m=\frac{1}{2}$. Upon the first supposition, we have

$$\frac{m}{m-1}=2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{\&c.}$$

and it has been shown (242), that the series, which constitutes the second member, approaches in fact, nearer and nearer to 2.

The second supposition leads us to

$$\frac{m}{m-1}=\frac{1}{0}=1+1+1+1+1+1+1\ \text{\&c.}$$

This result, $1+1+1+1+1$ &c. continued to infinity, presents in reality an infinite quantity, as the nature of the expression $\frac{1}{0}$ implies ; yet if we neglect the remainders in this example, we are led into an absurdity ; for since the divisor, multiplied by the quotient must produce the dividend, we have

$$1=(1+1+1+1+\qquad\qquad)\,0;$$

but the second member is strictly reduced to nothing, we have therefore $1=0$.

The third supposition leads to consequences not less absurd, if we neglect the remainders, and consider the series, which is obtained as expressing the value of the fraction from which it is derived.

Making $m = \frac{1}{2}$, we find

$$\frac{m}{m-1} = -1 = 1 + 2 + 4 + 8 + 16 + \&c.$$

which is evidently false.

There will be no contradiction of this kind, if we observe, that, in the second case, the remainders

$$1, \ \frac{1}{m}, \ \frac{1}{m^2}, \ \frac{1}{m^3}, \ \&c.$$

are each equal to 1, and that, since they do not diminish, they can never be neglected, to whatever extent the series is continued. If we add, therefore, one of these remainders to the second member of the equation

$$1 = (1 + 1 + 1 + 1 + 1 + \ . \ . \ . \ . \ . \ ) \, 0,$$

the equation becomes true. In the third case, the remainders,

$$1, \ \frac{1}{m}, \ \frac{1}{m^2}, \ \frac{1}{m^3}, \ \&c.$$

form the increasing progression, 1, 2, 4, 8, 16, &c. and, if we add to the several quotients the fractions, arising from the corresponding remainders, the exact expression for $\frac{m}{m-1}$ will be

$$1 + \frac{1}{m-1}$$

$$1 + \frac{1}{m} + \frac{1}{m(m-1)}$$

$$1 + \frac{1}{m} + \frac{1}{m^2} + \frac{1}{m^2(m-1)},$$

&c.

each of which gives -1, when $m = \frac{1}{2}$.

If we take $m = -n$, the fraction $\frac{m}{m-1}$ becomes $\frac{n}{n+1}$;

the series which is produced by developing this fraction, assumes the form

$$1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \&c.$$

and making $n = 1$, we have

$$1 - 1 + 1 - 1 + 1 - 1 + \&c.$$

a series, which becomes alternately 1 and 0, and which, consequently, as often exceeds, as it falls below, the true value of $\frac{n}{n+1}$, equal in this case to $\frac{1}{2}$; but as the above series is not converging, it cannot give this value; and we must, therefore, take into consideration the remainder, at whatever term we stop.

If we suppose, in the preceding series, $n = 2$, we have

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}, - \&c.$$

a series in which the particular sums, 1, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{4}$, &c. are, alternately, smaller and greater than the true value of $\frac{n}{n+1}$, which is $\frac{2}{3}$, but to which they approach continually, because the proposed series is converging.

Although *diverging* series, that is, those, the terms of which go on increasing, continue to depart further and further from the true value of the expressions from which they are derived, yet considered as developments of these expressions, they may serve to show such of their properties, as do not depend on their summation.

CCLXVI. If we continue any process of division in algebra, according to the method pursued above (243), with respect to the quantities m and $m - 1$, the quotient will always be expressed by an infinite series composed of *simple terms*. Infinite series are also formed by extracting the roots of imperfect powers, and continuing the operation upon the several successive remainders; but they are obtained more easily by means of the formula for binomial quantities.

Of Figurative Numbers.

CCLXVII. These numbers result from a single arithmetical progression, and each series of numbers is formed from it, by adding together the terms of the series which goes before, or each term is equal to its preceding added to the term immediately above it; they are called figurative numbers, because the units in each term may be so disposed as to represent a geometrical figure.

The following are figurative numbers.

1st order	1,	1,	1,	1,	1,	1,	1,	1,	1,	1	
2nd ..	1,	2,	3,	4,	5,	6,	7,	8,	9,	10	
3rd ..	1,	3,	6,	10,	15,	21,	28,	36,	45,	55	
4th	1,	4,	10,	20,	35,	56,	84,	120,	165,	220	
5th	1,	5,	15,	35,	70,	126,	210,	330,	495,	715	
6th	1,	6,	21,	56,	126,	252,	462,	792,	1287,	2002	
7th	1,	7,	28,	84,	210,	462,	924,	1716,	3003,	5005,	&c.

The 10th term of the 6th order, for example, is 2002 = sum of the ten first terms of the series of the 5th order, or the sum of its preceding term and that above it = 1287 + 715.

Without entering into the details of the summation of each order, we shall choose the series of the 3rd order as being the most useful to calculate the number of cannon shots in the triangular, square or oblong piles in which cannon shots are generally arranged in arsenals.

Let the numbers be disposed in the following order.

A

1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6

B

The sum of all the terms to the left of *A B*, viz. $1 + 3 + 6 + 10 + 15 = \frac{1}{3}$ of the sum of all the terms in the figure; for each vertical row on the right of *A B* is double the corresponding horizontal row on the left: thus $4 + 4 + 4 =$ twice $1 + 2 + 3$; $6 + 6 + 6 + 6 + 6 =$ twice $1 + 2 + 3 + 4 + 5$.

Now let n denote the number of horizontal ranks, then $n + 1$ will represent the number of terms in an horizontal rank or series, and also the last term of that series; the first plus the last term then $= 1 + (n + 1) = n + 2$ multiplied by $\frac{n+1}{2}$ or half the number of terms in the horizontal rank, will be the sum of the series $1 + 2 + 3 +$ &c., or of all the terms in that rank; multiplying this product by the number of horizontal ranks is $\frac{n+1}{2} \times (n + 2) \times n$, which is the sum of all the terms in the figure, and $\frac{1}{3}$ of that sum or $\frac{n+1}{2} \times (n + 2) \times n \times \frac{1}{3} = \frac{n}{1} . \frac{n+1}{2} . \frac{n+2}{3}$ is the sum of all the terms on the left of *A B*, or the sum of the series $1 + 3 + 6 + 10 +$ &c. continued to n terms.

CCLXVIII. Let the bottom row of a triangular pyramid consist of 9 shot, the proposed pile has then 9 courses, each of which forms an equilateral triangle, the shot contained in these, being in an arithmetical progression, of which the first and last term together with the number of terms are known; it follows then, that the sum of these courses or of the 9 progressions, will be the number of shot contained in the proposed pile.

The shot of the first or lowest triangular course		$(9 + 1) \times 4\frac{1}{2} =$	45
the second	..	$(8 + 1) \times 4 =$	36
the third	..	$(7 + 1) \times 3\frac{1}{2} =$	28
the fourth	..	$(6 + 1) \times 3 =$	21
the fifth	..	$(5 + 1) \times 2\frac{1}{2} =$	15
the sixth	..	$(4 + 1) \times 2 =$	10
the seventh	..	$(3 + 1) \times 1\frac{1}{2} =$	6
the eighth	..	$(2 + 1) \times 1 =$	3
the ninth	..	$(1 + 1) \times \frac{1}{2} =$	1

Sum $= 165$ shot, contained in the whole pile. The same number will be found by substituting 9 for n in the formula $\frac{n(n+1)(n+2)}{1 . 2 . 3}$, we have

thus $\frac{9 \times 10 \times 11}{6} = 165$. But if the number of shot were only, say, 50, it would be very tedious to find, first, the contents of all the courses, and then to add them.

Supposing then the number of courses in a triangular pile or pyramid were 50, we have $\frac{50}{1} \times \frac{50+1}{2} \times \frac{50+2}{4} = 22100$, the number of shots in such a pile.

The square pile is a pyramid, having a square for its base, and a single ball at the top; this ball, with the successive courses downward, constitute the series of squares, $1 + 4 + 9 + 16 +$ &c., the last term being the number of shot in the bottom course.

$$\begin{array}{ll} \text{The series of squares} & 1 + 4 + 9 + 16 + 25 + \ldots \text{ to } n \text{ terms.} \\ \text{may be resolved in} & \left\{ \begin{array}{l} 1 + 3 + 6 + 10 + 15 + \ldots \text{ to } n \text{ terms.} \\ \quad\;\; 1 + 3 + 6 + 10 + \ldots \text{ to } n-1 \text{ terms.} \end{array} \right. \\ \text{the two following} & \\ \text{Sum} & \overline{1 + 4 + 9 + 16 + 25 \text{ \&c.}} \end{array}$$

The first sum $1 + 3 + 6 + 10$ &c. to n terms is

$$\frac{n \times (n+1) \times (n+2)}{1 \;.\; 2 \;.\; 3},$$

and the 2nd, (substituting $n - 1$ for n) gives $\frac{n \times (n-1) \times (n+1)}{1 \;.\; 2 \;.\; 3}$ for the sum to $n - 1$ terms; adding both these expressions, we have

$$\frac{n \,.\, (n+1) \,.\, (n+2)}{1 \;.\; 2 \;.\; 3} + \frac{n \,.\, (n+1)\,(n-1)}{1 \;.\; 2 \;.\; 3} =$$

$$\frac{n \,.\, (n+1)}{1 \;.\; 2} \left[\frac{(n+2) + (n-1)}{3}\right] = \frac{n \,.\, (n+1) \,.\, (2n+1)}{1 \;.\; 2 \;.\; 3}$$

is the sum of the series of squares $1 + 4 + 9 + 16 + 25 +$ &c. continued to n terms.

The oblong pile stands on a rectangular base, consequently the number of shot in any course, is the product of the number of shot in one side by that in the other; the whole pile being composed of a series of rectangular courses, the side of each diminish by 1, from the base upwards; therefore, if d be the difference of the shot in the sides of any course, the pile will end at the top in a rank of $d + 1$ balls.

Let the sides of the bottom course contain 13 and 9 shot, the difference in the sides of each course will be 4.

The 1st course contains	$13 \times 9 =$	117
2nd	$12 \times 8 =$	96
3rd	$11 \times 7 =$	77
4th	$10 \times 6 =$	60
5th	$9 \times 5 =$	45
6th	$8 \times 4 =$	32
7th	$7 \times 3 =$	21
8th	$6 \times 2 =$	12
9th	$5 \times 1 =$	5
Number of shot in the whole pile	$=$	465

but the number of shot in an oblong pile can be found, by resolving the series $5 + 12 + 21 + 32 +$ &c. into two others. Thus

$$1 + 4 + 9 + 16 + 25 + \&c. = \frac{n . (n + 1)(2n + 1)}{1 . 2 . 3}$$

$$4 + 8 + 12 + 16 + 20 + \&c. = \frac{n (n + 1) d}{2}$$

$$\text{Sum } 5 + 12 + 21 + 32 + 45 + \&c. = \frac{n . (n + 1)(2n + 1)}{1 . 3 . 2}$$

$$+ \frac{n . (n + 1) d}{2} \text{ or } \frac{n . (n + 1)(2n + 1 + 3d)}{1 . 2 . 3}$$

is the sum of the series of products continued to n terms, where n is the number of courses, or the number of shot in the lesser side of the bottom course, and d the difference of the number of shot in the sides. Substituting, for the present example, 9 for n and 4 for d we have

$$\frac{9 \times 10 \times (18 + 1 + 12)}{1 . 2 . 3} = 465.$$

Should s be the number of shot in a complete triangular pile be given, and it were required to determine the number of shot in the side of the base; we have

$$s = \frac{n . (n + 1)(n + 2)}{1 . 2 . 3} \quad \text{or} \quad n^3 + 3n^2 + 2n = 6s$$

if to complete the cube of the first member, we add $n + 1$ to both members of the equation, we get

$$n^3 + 3n^2 + 3n + 1 = 6.s + n + 1 \quad \text{or}$$
$$n + 1 = \sqrt[3]{6.s + n + 1}$$

but as n must of necessity be a whole number, $\sqrt[3]{6s + n + 1}$ or $n + 1$ must be the integral cube next greater than $6s$.

Let the number of shot in a complete triangular pile be 20825; then $20825 \times 6 = 124950$, the cube next greater is 125000, whose root is $50 = n + 1 \quad \therefore \quad n = 49$.

In a complete square pile we have

$$s = \frac{n \,.\, (n+1)\,(2n+1)}{1 \,.\, 2 \,.\, 3} = \frac{2n^3 + 3n^2 + n}{6} \quad \text{or}$$

$$n^3 + 1\frac{1}{2}n^2 + \frac{1}{2}n = 3s$$

by adding $1\frac{1}{2}n^2 + 2\frac{1}{2}n + 1$, to both members of the equation, the 1st member becomes visibly a perfect cube, we get thus

$$n^3 + 3n^2 + 3n + 1 = 3s + 1\tfrac{1}{2}n^2 + 2\tfrac{1}{2}n + 1,$$

and by extracting the root

$$n + 1 = \sqrt[3]{3s + 1\tfrac{1}{2}n^2 + 2\tfrac{1}{2}n + 1}$$

$n + 1$ therefore, must be the root of the integral cube next greater than $3s$.

Suppose $s = 77531$, or $3s = 232593$, the cube next greater is 238328, whose cube root is $62 = n + 1$; wherefore $n = 61$.

Should the pile be incomplete, we find the number of shot in the whole pile, and the upper course of the broken pile being known, we deduct from the whole pile, the pile wanting; the difference is the number of shot in the incomplete pile.

In general, the figurative numbers of any order n, are found by substituting successively; 1 . 2 . 3 . 4 . 5 &c. instead of n, in the general expression

$$\frac{n \,.\, (n+1)\,(n+2)\,(n+3)\,(n+4)\,(n+5)\ \&c.}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6}$$

and to be continued till the number of factors is 1 less than that which expresses the rank of the figurative number. For example, let the 10th rank of the order $n = 7$, be required, taking 9 factors, we have

$$\frac{7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14 \times 15}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9} = 5005.$$

Examples of Progression by differences.

Let a represent the 1st term, z the last, δ the common difference, n the number of terms, and s the sum of the terms.

1. $a = 39$. $n = 157$. $\delta = \frac{1}{2}$ Reqs.
Ans. $s = 0$ if the series be decreasing, and $s = 12246$ is increasing.

2. $a = 1077\frac{5}{8}$. $n = 432$. $\delta = 5$. Required s. Ans. $s = 144$.

3. $a = \frac{1}{2}$. $n = 53$. $\delta = \frac{1}{4}$. Required s. Ans. $s = 371$.

4. $a = \frac{1}{10}$. $n = 124$. $\delta = \frac{1}{3}$. Required s. Ans. $s = 2554{,}4$.

5. $a = 5$. $n = 17$. $\delta = 12$. Required s. Ans. $s = 1717$.

6. $a = -\frac{5}{7}$. $n = 14986$. $\delta = \frac{4}{17}$ Required s. Ans. $s = 29935605\frac{5}{7}$.

7. $a = 6$. $n = 146$. $s = 127896$. Required δ. Ans. $\delta = 12$.

8. $a = 621$. $n = 425$. $s = 106250$. Required δ. Ans. $\delta = 1\frac{3}{4}$.

9. $a = 9042$. $n = 6745$. $s = 337250$. Required δ. Ans. $\delta = 2\frac{2}{3}$.

10. $a = 14$. $n = 445$. $z = 2900$. Required δ. Ans. $\delta = 6\frac{1}{3}$.

11. $a = 594$. $n = 229$. $z = 2000$. Required δ. Ans. $\delta = 6\frac{1}{8}$.

12. $a = 2048$. $n = 7993$. $z = 50000$. Required δ. Ans. $\delta = 6$.

13. $s = 7474$. $n = 101$. $\delta = 1{,}25$. Required a. Ans. $a = 11\frac{1}{3}$.

14. $s = 23320$. $n = 2332$. $\delta = -\frac{2}{7}$. Required a. Ans. $a = 343$.

15. $s = 24948$. $n = 99$. $z = 500$. Required a. Ans. $a = 4$.

16. $s = 298$. $n = 149$. $\delta = \frac{1}{4}$. Required a. Ans. $a = -16\frac{1}{3}$.

17. $s = 51005$. $n = 101$. $a = 5$. Required z. Ans. $z = 1005$.

18. $s = 92837$. $n = 731$. $a = 5\frac{1}{3}$ Required z. Ans. $z = 248\frac{2}{3}$.

19. $s = 116145$. $\delta = \frac{1}{3}$. $a = -17\frac{2}{3}$ Required n. Ans. $n = 890$.

20. $s = 97100$. $\delta = 2\frac{2}{3}$. $a = -1193\frac{1}{3}$. Required n. Ans. $n = 971$.

21. $s = 600$. $a = \frac{2}{5}$. $z = \frac{4}{5}$ Required n. Ans. $n = 1000$.

22. $s = 291640{,}5$. $a = \frac{1}{12}$. $z = 600$. Required n. Ans. $n = 972$.

Table of Formulæ of Progression by differences.

No.	Given.	Required.	Formulæ.
1	$a,\ \delta,\ n$		$= a + (n - 1)\,\delta$
2	$a,\ \delta,\ s$		$= -\frac{\delta}{2} \pm \sqrt{\left[2\,\delta\,s + \left(a - \frac{\delta}{2}\right)^2\right]}$
3	$a,\ n,\ s$	z	$= \frac{2\,s}{n} - a$
4	$\delta,\ n,\ s$		$= \frac{s}{n} + \frac{(n - 1)\,\delta}{2}$
5	$a,\ \delta,\ n$		$= \left[2\,a + (n - 1)\,\delta\right]\frac{n}{2}$
6	$a,\ \delta,\ z$		$= \frac{a + z}{2} + \frac{(z + a)\,(z - a)}{2\,\delta}$
7	$a,\ n,\ z$	s	$= \frac{a + z}{2}\,n$
8	$\delta,\ n,\ z$		$= \left[2\,z - (n - 1)\delta\right]\frac{n}{2}$
9	$a,\ n,\ z$		$= \frac{z - a}{n - 1}$
10	$a,\ n,\ s$		$= \frac{2\,(s - a\,n)}{n\,(n - 1)}$
11	$a,\ z,\ s$	δ	$= \frac{(z + a)\,(z - a)}{2\,s - z - a}$
12	$n,\ z,\ s$		$= \frac{2\,(n\,z - s)}{n\,(n - 1)}$
13	$\delta,\ n,\ z$		$= z - (n - 1)\,\delta$
14	$\delta,\ n,\ s$		$= \frac{s}{n} - \frac{(n - 1)\,\delta}{2}$
15	$\delta,\ z,\ s$	a	$= \frac{\delta}{2} \pm \sqrt{\left[\left(z + \frac{\delta}{2}\right)^2 - 2\,\delta\,s\right]}$
16	$n,\ z,\ s$		$= \frac{2\,s}{n} - z$
17	$a,\ \delta,\ z$		$= 1 + \frac{z - a}{\delta}$
18	$a,\ \delta,\ s$	n	$= \frac{\delta - 2\,a \pm \sqrt{\left[8\,s\,\delta + (2\,a - \delta)^2\right]}}{2\,\delta}$
19	$a,\ z,\ s$		$= \frac{2\,s}{a + z}$
20	$\delta,\ z,\ s$		$= \frac{2\,z + \delta \pm \sqrt{\left[(2\,z + \delta)^2 - 8\,\delta\,s\right]}}{2\,\delta}$

Table of Formulæ of Progression by Quotients.

No.	Given.	Required.	Formulæ.
1	a, q, n	z	$= a\, q^{n-1}$
2	a, q, s		$= \dfrac{a + (q-1)\, s}{q}$
3	a, n, s		$z\,(s - z)^{n-1} = (s - a)^{n-1}$
4	q, n, s		$= \dfrac{(q-1)\, s\, q^{n-1}}{q^n - 1}$
5	a, q, n	s	$= \dfrac{a\,(q^n - 1)}{q - 1}$
6	a, q, z		$= \dfrac{q\, z - a}{q - 1}$
7	a, n, z		$= \ldots \dfrac{z^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{z^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$
8	q, n, z		$= \dfrac{z\,(q^n - 1)}{(q-1)\, q^{n-1}}$
9	a, n, z	q	$= \sqrt[n-1]{\dfrac{z}{a}}$
10	a, n, s		$\dfrac{s}{a}\, q - q^n = \dfrac{s - a}{a}$
11	a, z, s		$= \dfrac{s - a}{s - z}$
12	n, z, s		$\left(\dfrac{s}{s - z}\right) q^{n-1} - q^n = \dfrac{z}{s - z}$
13	q, n, z	a	$= \dfrac{z}{q^{n-1}}$
14	q, n, s		$= \dfrac{(q-1)\, s}{q^n - 1}$
15	q, z, s		$= q\, z - (q - 1)\, s$
16	n, z, s		$z\,(s - z)^{n-1} = a\,(s - a)^{n-1}$
17	a, q, z	n	$= \dfrac{log.\, z - log.\, a}{log.\, q} + 1$
18	a, q, s		$= \dfrac{log.\,[a + (q-1)\, s] - log.\, a}{log.\, q}$
19	a, z, s		$= \dfrac{log\, z - log\, a}{log\,(s - a) - log\,(s - z)} + 1$
20	q, z, s		$= \dfrac{log\, z - log\,[q\, z - (q-1)\, s]}{log\, q} + 1$

Examples in Progression by differences.

1. *Required the sum of an increasing arithmetical series, having* 2 *for its first term,* 3 *for the common difference, and* 31 *for the number of terms.*

Ans. 1457.

2. *Required the sum of a decreasing arithmetical progression having* 12 *for its first term,* $\frac{1}{5}$ *for the common difference, and* 25 *or the number of terms.*

Ans. 240.

3. *The dials of astronomical clocks are generally marked to* 24 *hours, what is the sum of all the figures on the dial gone over by the minute hand during the month of August?*

Ans. 223200.

4. *One hundred and twenty-one eggs, being placed on the ground, in a straight line, at the distance of a yard and a half from each other, how far will a man have travelled who brought them one by one to a basket, placed at one and a half yard distant from the first egg?*

Ans. 12 *miles and* 1023 *yards.*

5. *Required the sum of all the odd numbers continued up to* 4000.

Ans. 4000000*.

6. *What is the* 37*th term of the series* 12, $11\frac{8}{9}$, $11\frac{2}{9}$, $11\frac{2}{3}$ *&c.*

Ans. 8.

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* In this progression the 1st term is 1, the last $1 + (n - 1)\,d$, or because $d = 2$, $1 + (n - 1)\,2$, therefore the sum $= [2 + (n - 1)\,2]\,\frac{n}{2} = (2 + 2n - 2)\,\frac{n}{2} = n^2$. It appears then that the sum of the odd numbers continued to n terms, is n^2. Also, the sum of any two consecutive terms of the series of the triangular numbers 1 . 3 . 6 . 10 . 15 . 21, &c. is always a square; for $\frac{(n-1)\,n}{2} + \frac{(n+1)\,n}{2} = n^2$.

7. *What is the difference between all the even, and all the odd terms of the series of the natural numbers* 1, 2, 3, 4, *&c. to* 900 *terms?*

Ans. 450.

8. *Two detachments, of opposite armies, distant from each other* 39 *leagues, both designing to occupy an advantageous post equidistant from each other's camp, set out at different times; the first detachment increasing every day's march one league and a half, and the second detachment decreasing each day's march two leagues; both detachments arrived at the same time; the first after* 5 *days' march, and the second after* 4 *days' march. What is the number of leagues marched by each detachment each day?*

Ans. *the* 1*st detachment* $\frac{9}{10}$, $2\frac{4}{10}$, $3\frac{9}{10}$, $5\frac{4}{10}$, $6\frac{9}{10}$

2*nd* $7\frac{7}{8}$, $5\frac{7}{8}$, $3\frac{7}{8}$, $1\frac{7}{8}$.

9. *A bailiff is sent after a man, who had made his escape from jail, who runs* 60 *miles the first day, but as his strength began to fail him, he was obliged to diminish his speed by* 10 *miles every day; the bailiff continuing to march every day* 35 *miles: in how many days did he overtake him?*

Ans. 6 *days.*

10. *How many terms are in a decreasing arithmetical series, of which the first term is* 16, *the common difference* $\frac{2}{21}$, *and the sum of all the terms* 602.

Ans. 43.

11. *The first term of an increasing arithmetical series is* — 81 *the common difference* $\frac{3}{7}$ *and the sum of all the terms is* 87216; *what are the number of terms?*

Ans. 87210.

12. *A company of foot leave Calcutta for Bettiah, a distance of* 374 *miles, and at the same time a party of horse are ordered from Bettiah to Calcutta; the foot march* 16 *miles the first day,* $15\frac{1}{3}$ *the second, lessening each day's*

march by $\frac{2}{3}$ of a mile; but the horse travel eight miles and a half the first day, and increase their march one and half mile every day; what distance will each party have travelled when they meet?

The horse, 220,1 *miles.*
The foot, 153,9 *ditto.*

13. *Supposing that in the preceding question, the company of horse were already four days on their march, when that of foot set out, (all other circumstances remaining the same,) how many miles would each party have travelled when they meet?*

The horse 247 *miles.*
The foot 127 *ditto.*

14. *The sum of Rs.* 27, *was to be raised by subscription by three persons,* A, B, *and* C; *the sums to be subscribed by them respectively forming an arithmetical progression. But* C *dying before the money was paid, the whole fell to* A *and* B; *and* C's *share was raised between them in the proportion of* 3 : 2, *when it appeared that the whole sum subscribed by* A *was to the whole sum subscribed by* B : : 4 : 5. *Required the original subscription of* A, B, *and* C.

Ans. 3, 9 *and* 15.

15. *I had a cook whom I engaged for* 2 *rupees* 4 *annas per month; as my means were improving, I increased his wages every month by* 12 *annas; remaining with me* 10 *years and* 7 *months, how much did he receive the last month?*

Ans. Rs. 9. 8 *ans.*

16. *A debtor, unable to pay at once his debt amounting to Rs.* 73206, *agrees with his creditor to discharge it by monthly instalments, on condition that he should not be charged with any interest, viz. the* 1*st month Rs.* 390, *but each succeeding month Rs.* 12, *more than the preceding. In what time will he have discharged his whole debt? And how much does he pay the last month?*

Ans. 6 *years,* 11 *months;*
last month Rs. 1374.

17. *A company of seapoys who had successfully stormed a fortress, was rewarded as follows: that seapoy who had mounted the wall first, received a certain sum of money, the second some rupees less, the third exactly so much less than the second, and so on. When the money was divided, two of*

the soldiers could not be present on account of their wounds; their share was then given to two of their comrades, these two put both their own money, and that of their comrades into the same purse, and when they came to divide it with them, they had forgotten what fell to each man's share. One had received Rs. 502 *for himself and comrade, and only remembered, that he himself was the* 15*th, and his comrade the* 11*th, the other had received Rs.* 670, *for himself and his comrade, and knew that he was the* 2*nd, and his comrade the* 10*th. How much did each of the four soldiers receive?*

The 2*nd Rs.* 383, *the* 10*th Rs.* 287.
The 11*th Rs.* 275, *the* 15*th Rs.* 227.

18. *The base of a right-angled triangle is* 12 *feet, and the three sides are in arithmetical progression; it is required to find the length of the hypothenuse?*

Ans. 15 *feet.*

19. A *is dispatched to Chandanagor, travelling* 3 *miles the first hour, but relaxing his speed* $\frac{1}{3}$ *of a mile every hour; one hour and a quarter after,* B *is sent to overtake him, going only* 3 *miles the first hour, but gaining* $\frac{8}{21}$ *of a mile every hour. In how many hours will he overtake him?*

Ans. $5\frac{3}{4}$ *hours.*

20. *Given the sum of the squares of the two extremes* = 477, *and the sum of the squares of the two means* = 377; *to determine the four numbers.*

Ans. 6, 11, 16, 21.

21. *In an arsenal there is a triangular pyramid of which the side of the lowest layer has* 101 *cannon balls; how many balls are there in the whole pyramid?*

Ans. 176851.

22. *A triangular pyramid being incomplete, the side of the lowest layer containing* n *balls, and the side of the highest layer* m *balls. Give the general expression for the number of balls in a broken pyramid?*

$$Ans.\quad \frac{n\,(n+1)\,(n+2)}{1\,.\,2\,.\,3} - \frac{m\,(m+1)\,(m-1)}{1\,.\,2\,.\,3}.$$

23. *In a quadrilateral incomplete pyramid each side of the lowest layer contains* n *and each side of the highest* m *balls.*

Express, by a general equation, the number of balls in an incomplete quadrilateral pyramid?

$$Ans. \quad \frac{n}{1} \cdot \frac{(n+1)}{2} \cdot \frac{(2n+1)}{3} - \frac{m}{1} \cdot \frac{(m-1)}{2} \cdot \frac{(2m-1)}{3}$$

24. *In a broken rectangular pile. of which the sides in the bottom course contain* 35 *and* 17 *and the greater side of the upper course* 24; *what number of shots are contained in the broken pile?*

Ans. 4214.

25. *In an incomplete rectangular pile, the sides of the lowest layer has* n *and* n + d *shots, the less side of the highest layer contains* m *shots. What is the general expression of the whole number of shots contained in the incomplete pile?*

$$Ans. \quad \frac{n}{1} \cdot \frac{(n+1)}{2} \cdot \frac{(2n+1+3d)}{3} \cdot \frac{m}{1} \cdot \frac{(m-1)}{2} \cdot \frac{(2m-1+3d)}{3}$$

26. *The number of cannon balls in a triangular pyramid amounts to* 23426. *How many balls are contained in the side of the bottom course?*

Ans. 51.

27. 45526 *shots piled form a complete quadrilateral pyramid? How many shots are in a side of the bottom course?*

Ans. 51.

28. *An oblong pile of shells, of which the difference in the sides of the bottom course is* 11 *shells, contains* 60112 *shells? What must be the number of shells in the less side of the bottom course?*

Ans. 51.

29. *In a complete rectangular pile, there are* 30340 *shells, and the difference of the sides is* 10. *Determine the number of shells in the lowest layer.*

Ans. 400.

30. *Having occasion to take away the uppermost five layers from the pyramid in the last example, how many shells are in the greater side of the highest course.*

Ans. 16.

31. *In an incomplete oblong pile, containing* 4640 *shot, the difference of the sides is* 9. *What is the number of shot in the greater side of the lowest course?*

Ans. 29.

Examples in Progression by Quotients.

Let q represent the common ratio or quotient, ∞ infinity, a, z, n, s as in page 373.

	Given Values.			Required Values.	
1	$a = 1$;	$q = 2$;	$n = 7$	$z = 64$;	$s = 127$
2	$a = 4$;	$q = 3$;	$n = 10$	$z = 78732$;	$s = 118096$
3	$a = \frac{1}{100}$;	$q = 8$;	$s = 1533916,89$	$z = 1342177,28$;	$n = 10$
4	$a = \frac{3}{10}$;	$q = \frac{1}{10}$;	$s = \frac{1}{3}$	$z = 0$;	$n = \infty$
5	$a = \frac{1}{7}$;	$n = \infty$;	$s = \frac{1}{7}$	$z = 0$;	$q = \frac{1}{10}$
6	$a = \frac{1}{125}$;	$n = 17$;	$s = 381469726,5625$	$z = 1220703125$;	$q = 5$
7	$q = \frac{\beta}{\gamma}$;	$n = \infty$;	$s = \frac{\alpha\gamma}{\gamma - \beta}$	$z = 0$;	$a = \alpha$
8	$q = 2$;	$n = 32$;	$s =$ £4473924.5*s.* 3¾*d.*	. . . . . .	$a =$ 1 farthing.
9	$a = \frac{1}{128}$;	$q = 32$;	$z = 8796093022208$	. . . . . .	$n = 10$
10	$a = 5$;	$q = 4$;	$z = 327680$	$s = 436905$;	$n = 9$
11	$a = 3$;	$n = 25$;	$z = 9642,612$	$s = 38741,59...$;	$q = \frac{5}{7}$

	Given Values.			Required Values.	
12	$a = 40$;	$n = \infty$;	$z = 0$	$s = 70$;	$q = \frac{5}{7}$
13	$q = 4$;	$n = 12$;	$z = 838860\frac{4}{5}$	$s = 1118481$;	$a = \frac{1}{5}$
14	$q = 2$;	$n = 12$;	$z = 2048$	$s = 4095$;	$a = 1$
15	$q = 9$;	$z = 2657205$;	$s = 2989355$	$a = 5$;	$n = 7$
16	$q = 5$;	$z = 11718750$;	$s = 14648436$	$a = 6$;	$n = 10$
17	$n = 16$;	$z = \frac{98304}{14348907}$;	$s = 8\frac{42456897}{63046721}$	$a = 3$;	$q = \frac{2}{3}$
18	$n = 6$;	$z = 46656$;	$s = 55986$	$a = 6$;	$q = 6$
19	$a = 7$;	$z = \frac{15309}{16384}$;	$s = 25\frac{3225}{16384}$	$q = \frac{3}{4}$;	$n = 8$
20	$a = \frac{2}{3}$;	$z = \frac{1}{12288}$;	$s = 1\frac{4095}{12288}$	$q = \frac{1}{2}$;	$n = 14$
21	$a = 1$;	$z = 0$;	$n = \infty$	$s = 2$;	$q = \frac{1}{2}$
22	$a = 0{,}2$;	$q = 4$;	$n = 8$	$s = 4369$;	$z = 3276{,}8$
23	$a = \frac{1}{35}$;	$s = 16120628042\frac{3}{5}$;	$z = 134338567026\frac{6}{7}$	$q = 6$;	$n = 16$
24	$s = 313842837671$;	$q = 11$;	$z = 285311670611$	$a = 11$;	$n = 11$
25	$s = 466033\frac{2}{3}$;	$q = 4$;	$a = \frac{1}{3}$	$z = 349525\frac{1}{3}$;	$n = 11$
26	$s = 3{,}07861328$;	$q = 1{,}5$;	$a = \frac{1}{16}$	$z = 10678710\frac{15}{16}$;	$n = 8$

Examples in Progression by Quotients.

1. *There are three numbers in geometrical progression, the greatest of which exceeds the least by* 56. *Also the sum of the squares of the greatest and least is to the sum of the squares of all the three numbers, as* 82 : 91. *Required the numbers.*

Ans. 7, 21 *and* 63.

2. *The sum of three numbers in geometrical progression is* $\frac{13}{3}$, *and the product of the mean by the sum of the extremes is* $\frac{10}{3}$ *What are the numbers?*

Ans. $\frac{1}{3}$, 1, *and* 3.

3. *There is a number consisting of* 4 *digits; the sum of the* 1*st and* 2*nd digits, is to the sum of the* 2*nd and* 3*rd, as the sum of the* 1*st and* 4*th digits, is to the sum of the* 2*nd and* 4*th; and the square of the first digit, is to the square of the second increased by* 2, *as the last digit, is to the first; again, the number itself diminished by* 3, *is to the sum of the four digits, as* 223 : 1. *Lastly, by subtracting* 1818 *from the number the digits will be inverted. Find the number.*

Ans. 3571.

4. *There are three numbers in geometrical progression, the greatest of which exceeds the least by* 18. *Also the sum of the squares of the mean and least, is to the sum of the squares of all the three numbers as* 5 : 21. *Required the numbers.*

Ans. 24, 12, 6.

5. *The sum of three numbers in geometrical progression is* 26, *and the mean multiplied by the sum of the extremes is* 120. *Required the numbers.*

Ans. 2, 6, 18.

6. *There are five numbers, the three first of which are in geometric progression; the three last in arithmetic progression, the first number being half the difference of the last two numbers. The sum of the four first* = 72, *and the product of the first by the last* = 117. *What are the numbers?*

Ans. 3, 9, 27, 33, 39.

7. *There are five numbers in geometric progression, the difference of the squares of the last and the sum of the three first terms is equal to* 785. *The sum of the mean term and the cube of the square root of the first term, the sum of the fourth and mean term, the last term diminished by* 1, *the sum of the fourth and first term, and the sum of the mean and second term form five numbers in a decreasing arithmetic progression of which the common difference is* 10. *Required the numbers.*

Ans. 16, 24, 36, 54, 81.

8. *What is the sum of the following series of* n*th powers continued to* t *terms,* $a^n + a^n r^n + a^n r^{2n} + a^n r^{3n} +$ &c.

Ans. $\dfrac{a^n (r^{nt} - 1)}{r^n - 1}$.

9. *Required the sum of the descending series* $1 - x + x^2 - x^3 + x^4 -$ &c. *to an infinite number of terms.*

Ans. $\dfrac{1}{1+x}$.

10. *What is the difference of the two series* $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8}$ *ad infinitum, and* $\dfrac{8}{27} + \dfrac{4}{9} + \dfrac{2}{3}$ *to four terms?*

Ans. $\dfrac{11}{65}$.

Geometry teaches that the volumes of spheres, are in the ratio of the cubes of their radii; we can then find the weight of any ball or shell, knowing that of a ball of the same metal of any diameter whatever, and if the specific gravity is also given, the weight of any sphere, whatever its composition, can be determined.

11. *A cannon ball of cast iron* $3\frac{1}{2}$ *inches diameter, weighs* 5,792 *lbs.; what will one of the same metal of* 4, 5 *inches weigh?*

Ans. 12,310.

12. *Required the weight of a shell, of the same metal, whose radius is* 3, 1 *inches and the thickness of the metal* 1, 4 *inch.*

Ans. 26,886.

13. *Required the weight of a sphere* 3, 4 *inches in diameter of a sort of wood whose specific gravity is to that of cast iron as* 1 : 14, 5.

Ans. 5, 8187 *ounces.*

14. *What must be the thickness of a shell of fine gold, five inches radius, so that only half of it will be immersed, if it is made to float in fresh water?*

Ans. .0433 *inch.*

15. *A sphere of silver whose radius is* r, *is to be cast into two balls whose radii must be as* m : n. *Express these radii.*

Ans. $\dfrac{m\,r}{(m^3 + n^3)^{\frac{1}{3}}}$ *and* $\dfrac{n\,r}{(m^3 + n^3)^{\frac{1}{3}}}$.

16. *The ratio of the volumes of a cylinder : sphere : cone : :* 1 : $\frac{2}{3}$: $\frac{1}{3}$, *if the three have the same height and the weight of a cone be* 14; *what is that of the other two?*

Ans. $9\frac{1}{3}$; $4\frac{2}{3}$.

17. *It is demonstrated in the Elements of Mechanics: that the density of a material body, is as the mass directly, and as the volume inversely. The mass of the sun, being 354936, and its volume 1384472 greater than that of the earth, what must be the sun's density?*

Ans. 0,2543.

18. *The attraction of spheres is proportional to the masses directly and the square of the distance inversely, the radius of the earth being* 4000, *and that of the sun* 440000 *miles (in round numbers), required the intensity of solar gravity, that of the earth being* 1.

Ans. 27, 9.

19. *If a mass weigh* 170 *lbs. on the earth, what pressure will it have at the surface of the sun?*

Ans. 4600. *lbs.*

20. *The distance of the moon from the earth's centre is (somewhat less than) sixty times the distance from the centre to the surface, where gravity is estimated at* $32\frac{1}{6}$ *feet per second, What is the period of the revolution of the moon?*

Ans. 27 *days,* 7*h.* 43*m.*

21. *The quantity of matter of the moon, is to that of the earth as* 1 : 39, *and taking the distance of their centres as* 240000 *miles; at what distance from the moon in the line joining their centres, would a body be equally attracted by the earth and moon? (See ex.* 18.)

Ans. $\dfrac{240000}{1+\sqrt{39}} = 33126$ *miles nearly.*

22. *To what height above the earth's surface must a body be transported, to lose* $\frac{1}{12}$ *of its weight?*

Ans. 176,22 *miles.*

23. *How far beneath the earth's surface must a body be, to lose* $\frac{1}{12}$ *of its weight. The force of gravity being, in this case, directly as the distance?*

Ans. $33\frac{1}{3}$ *miles.*

24. *A soldier escaping from a fortress, ran the* 1*st hour* $9\frac{1}{2}$ *miles, the second hour* 9 *miles, and so on, lessening his speed by* $\frac{1}{2}$ *mile every hour. Five hours after, his escape being detected, a detachment of dragoons were sent after him; not knowing precisely the road the deserter had taken, they approached him (by computation made afterwards), only,* $1\frac{2}{8}\frac{5}{8}$ *miles the first hour; but as*

by enquiry, they were made more certain about the road they had to follow, neared the deserter every hour, and it was found that they did so in a geometrical progression, the ratio of which was 2. How many hours after the deserter's escape, was he apprehended?

Ans. 11 *hours.*

Students acquainted with the elementary parts of mechanics, hydrostatics, maxima and minima questions, astronomy, and trigonometry, will find, amongst the miscellaneous questions at the end of the present volume, where no reference is made to any particular law, many interesting questions involving geometrical progressions.

Theory of Exponential Quantities and of Logarithms.

CCLXIX. In the several questions we have resolved thus far, the unknown quantities have not been made subjects of consideration as exponents; this will be requisite, however, if we would determine the number of terms in a progression by quotients, of which the first term, the last term, and the ratio are given. In fact, we are furnished by a question of this kind with the equation

$$z = a\, q^{n-1} \text{ (260)},$$

in which n will be the unknown quantity; abridging the expression, by making $n - 1 = x$, we have $z = a\, q^x$. This equation cannot be resolved by the direct methods hitherto explained; and quantities like x cannot be represented by any of the signs already employed. In order to present this subject in a more clear light, we shall go back to state, according to Euler, the connexion which exists between the several algebraic operations, and the manner, in which they give rise to new species of quantities.

CCLXX. Let a and b be two quantities, which it is required to add together; we have

$$a + b = c;$$

and in seeking a or b from this equation, we find

$$a = b - c, \quad b = c - a;$$

hence the origin of subtraction; but when this last operation cannot be performed in the order in which it is indicated, the result becomes negative.

The repeated addition of the same quantity gives rise to multiplication; a representing the multiplier, b the multiplicand, and c the product, we have

$$a\,b = c,$$

whence we obtain

$$a = \frac{c}{b},\ b = \frac{c}{a};$$

and hence arises division, and fractions, in which this division terminates, when it cannot be performed without a remainder.

The repeated multiplication of a quantity by itself produces the powers of this quantity: if λ represent the number of times a is a factor in the power under consideration, we have

$$a^{\lambda} = c.$$

This equation differs essentially from the preceding, as the quantities a and λ do not both enter into it of the same form, and hence the equation cannot be resolved in the same way with respect to both. If it be required to find a, it may be obtained by simply extracting the root, and this operation gives rise to a new species of quantities, denominated irrational; but λ must be determined by peculiar methods, which we shall proceed to illustrate, after having explained the leading properties of the equation $a^{\lambda} = c$.

CCLXXI. It is evident, that if we assign a constant value greater than unity to a, and suppose that of λ to vary, as may be requisite, we may obtain successively for c all possible numbers. Making $\lambda = 0$, we have $c = 1$; then since λ increases, the corresponding values of c will exceed unity more and more, and may be rendered as great as we please. The contrary will be the case, if we suppose λ negative; the equation $a^{\lambda} = c$ being then changed into $a^{-\lambda} = c$, or $\frac{1}{a^{\lambda}} = c$, the values of c will go on decreasing, and may be rendered indefinitely small. We may, therefore, obtain from the same equation all possible positive numbers, whether entire or fractional, upon the supposition, that a exceeds unity. The same is true, if we have $a < 1$; only the order in which the values stand, will be reversed; but if we suppose $a = 1$, we shall always find $c = 1$, whatever value be assigned to λ; we must, therefore, consider the observations which follow, as applying only to cases, in which a differs essentially from unity.

In order to express more clearly, that a has a constant value, and that the two other quantities λ and c are indeterminate, we

shall represent them by the letters x and y; we then have the equation $a^x = y$, in which each value of y answers to one value of x, so that either of these quantities may be determined by means of the other.

CCLXXII. This fact, that all numbers may be produced by means of the powers of one, is very interesting, not only when considered in relation to algebra, but also on account of the facility with which it enables us to abridge numerical calculations. Indeed, if we take another number y', and designate by x' the corresponding value of x, we shall have $a^{x'} = y'$, and consequently, if we multiply y by y', we have

$$y\,y' = a^x \times a^{x'} = a^{x+x'};$$

if we divide the same, the one by the other, we find

$$\frac{y'}{y} = \frac{a^{x'}}{a^x} = a^{x'-x};$$

lastly, if we take the m^{th} power of y, and the n^{th} root, we have

$$y^m = (a^x)^m = a^{mx}$$

for the one, and

$$y^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}$$

for the other.

It follows from the first two results, that knowing the exponents x and x' belonging to the numbers y and y', we may, by taking their sum, find the exponent which answers to the product $y\,y'$, and by taking their difference, that which answers to the quotient $\frac{y'}{y}$. From the last two equations it is evident, that the exponent belonging to the m^{th} power of y, may be obtained by simple multiplication, and that which answers to the n^{th} root, by simple division.

Hence it is obvious, that by means of a table, in which, against the several numbers y, are placed the corresponding values of x, y being given, we may find x, and the reverse; and the *multiplication of any two numbers is reduced to simple addition*, because, instead of employing these numbers in the operation, we may add the corresponding values of x, and then seeking in the table the number, to which the sum answers, we obtain the product required. The quotient of the proposed numbers may be found, in the same table, opposite the difference between the corresponding values of x, and, therefore, *division is performed by means of subtraction.*

These two examples will be sufficient to enable us to form an idea of the utility of tables of the kind here described, which have been applied to many other purposes since the time of Napier, by whom they were invented. The values of x are termed *logarithms*, and, consequently, *logarithms are the exponents of the powers, to which a constant number must be raised, in order that all possible numbers may be successively deduced from it.*

The constant number is called the base of the table or system of logarithms.

Representing the logarithm of y by $l\,y$; we have then $x = l\,y$, and taking $y = b^x$, we obtain the equation $y = b^{ly}$.

CCLXXIII. As the properties of logarithms are independent of any particular value of the number b, or of their base, we may form an infinite variety of different tables by giving to this number all possible values, except unity. Taking, for example, $b = 10$, we have $y = (10)^{ly}$, and we discover at once, that the numbers

1, 10, 100, 1000, 10000, 100000, &c.

which are all powers of 10, have, for logarithms, the numbers

0, 1, 2, 3, 4, 5, &c.

The properties mentioned in the preceding article, may be verified in this series; thus, if we add together the logarithms of 10 and 1000, which are 1 and 3, we perceive, that their sum, 4, is found directly under 10000, which is the product of the proposed numbers.

CCLXXIV. Before proceeding to find the logarithm of the numbers intermediate between 1 and 10, 10 and 100, 100 and 1000, &c. which can only be obtained by approximation, it is necessary to introduce the theory of

Recurring Series.

It has been shown (art. 260), that the progression by the same ratio, or the progression of quotients

$$a + a\,q + a\,q^2 + a\,q^3 + \text{\&c.}$$

arises from the development of the fraction $\dfrac{a}{1-q}$; this circumstance naturally leads us to examine the series which may result from the development of any fraction whatever. Let the

fraction be $\frac{a}{b + c x}$, $= a\,(b + c\,x)^{-1}$, whence

$$\frac{a}{b + c\,x} = A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + F\,x^5 + \&c.$$

where the letters A, B, C, &c. are indeterminate.

CCLXXV. Let the following general equation be proposed, which for the sake of reference, we call

$$(P) \quad . \quad . \quad . \quad m = a\,y + b\,y^2 + c\,y^3 + d\,y^4 + \&c.$$

in this equation the value of m as well as that of the coefficients a, b, c, d, &c. are known; y is the unknown quantity whose value is to be ascertained.

CCLXXVI. Let the series be converted into the following, which call

$$(Q) \quad . \quad . \quad . \quad y = A\,m + B\,m^2 + C\,m^3 + D\,m^4 + \&c.$$

It is evident, that we can always represent thus the value of y, as A, B, C, D, &c. are *indeterminate* expressions and consequently each susceptible of any value whatever. It becomes therefore necessary to ascertain the value of each of these indeterminate quantities. Nothing is easier, and at the same time more ingenious.

CCLXXVII. As $y = A\,m + B\,m^2 + C\,m^3 + \&c.$ let us take successively the 1st, the 2nd, the 3rd, the 4th, &c. powers of this equation: let these powers be arranged according to the different powers of m, we have then

$$\begin{array}{llllll}
y = A\,m + & B\,m^2 + & C\,m^3 + & D\,m^4 + & E\,m^5 + \&c. \\
y^2 = & A^2m^2 + & 2\,A\,B\,m^3 + & 2\,A\,C\,m^4 + & 2\,A\,D\,m^5 + \&c. \\
 & & & B^2m^4 + & 2\,B\,C\,m^5 + \&c. \\
y^3 = & & A^3\,m^3 + & 3\,A^2B\,m^4 + & 3\,A^2C\,m^5 + \&c. \\
 & & & & +\,3\,A\,B^2m^5 + \&c. \\
y^4 = & & & A^4m^4 + & 4\,A^3B\,m^5 + \&c. \\
y^5 = & & & & A^5m^5 + \&c. \\
\&c. & & & &
\end{array}$$

CCLXXVIII. Let now these values of y, y^2 y^3 &c. be substituted in the equation (P), by disposing the terms in the same order, and writing, for shortness' sake, the powers of m only once at the head of each column, and imagining them repeated in the terms under it, thus:

$$m = \left\{ \begin{array}{lllllll} a\,y = aAm & + aBm^2 + & aCm^3 + & aDm^4 + & aEm^5 & +\&c. \\ +b\,y^2 = & \ldots & bA^2 & +2ABb & +2ACb & +2ADb & +\&c. \\ & & & & + B^2b & +2BCb & +\&c. \\ +c\,y^3 = & \ldots & \ldots & A^3\,c & +3A^2Bc & +3A^2Cc & +\&c. \\ & & & & & +3AB^2c & +\&c. \\ +d\,y^4 = & \ldots & \ldots & \ldots & A^4d & +4A^3Bd & +\&c. \\ +e\,y^5 = & \ldots & \ldots & \ldots & \ldots & A^5e & +\&c. \\ + \&c. & & & & & & \end{array} \right.$$

and in transposing m into the second member of the equation, it becomes $= 0$ or $0 = m\,(a\,A - 1) + m^2\,(a\,B + b\,A^2) + m^3\,(C\,a + 2\,A\,B\,b + A^3\,c) + \&c.$

CCLXXIX. The values then of A, B, C, &c. must be such, that the second member of this fraction, however far produced, be always equal to zero, for if the above equation be equal to zero, by adding a new column beginning with $a\,F\,m^6 +$ &c. it will still be equal to zero, but this condition can only hold, when the values of the indeterminates are such, that each term of the equation $0 = m\,(a\,A - 1) + m^2\,(a\,B + b\,A^2) +$ &c. is equal to zero*. We have thus $0 = m\,(a\,A - 1)$ or $m\,a\,A = m$ or $a\,A = 1$ or $A = \dfrac{1}{a}$; again, $m^2\,(a\,B + b\,A^2) = 0 \therefore B = -\dfrac{b\,A^2}{a}$;

or, by substituting for A its value found, $B = -\dfrac{b}{a^3}$. In the same way we find

$$C = \frac{2\,b^2 - a\,c}{a^5},\quad D = \frac{5\,a\,b\,c - a^2d - 5\,b^3}{a^7};$$

$$E = \frac{14\,b^4 - 21\,a\,b^2\,c + 6\,a^2\,b\,d + 3\,a^2\,c^2 - a^3\,e}{a^9}, \text{ \&c.}$$

Substituting these values in equation (Q), there remains no unknown quantity in the second member, the value of the quantity sought y is found as follows:

$$y = \frac{m}{a} - \frac{b\,m^2}{a^3} + (2\,b^2 - a\,c)\frac{m^3}{a^5} + (-5\,b^3 + 5\,a\,b\,c - a^2\,d)\frac{m^4}{a^7}$$

$$+ (14\,b^4 - 21\,a\,b^2\,c + 6\,a^2\,b\,d + 3\,a^2c^2 - a^3\,e)\frac{m^5}{a^9} +$$

---

* If $\alpha\,x + \beta\,x^2 + \gamma\,x^3 + \delta\,x^4 +$ &c. $= 0$, whatever value may be assigned to x, &c. and however far the series be continued, then the coefficients α, β, γ, &c. must each be $= 0$.

$(-42\,b^5 + 84\,a\,b^3\,c - 28\,a^2\,b^2\,d - 28\,a^2\,b\,c^2 + 7\,a^3\,b\,e + 7\,a^3\,c\,d - a^4\,f)\,\frac{m^6}{a^{11}} + \&c.$

CCLXXX. This series will serve as a general formula for converting any series of whatever form. Let there be given the equation

$$(S)\quad A = \sin.\,A + \frac{\sin.^3 A}{2\,.\,3} + \frac{3\,\sin.^5 A}{2\,.\,4\,.\,5} + \frac{3\,.\,5\,\sin.^7 A}{2\,.\,4\,.\,6\,.\,7} + \frac{3\,.\,5\,.\,7\,\sin.^9 A}{2\,.\,4\,.\,6\,.\,8\,.\,9} + \&c.$$

comparing it with the general equation (P), it will be observed that it contains no even powers, yet the equation (P) and that to be resolved must be similar. They become so when the coefficients of those powers which are wanting in equation (S), are equalised to zero; i. e. when $b = 0$, $d = 0$, $f = 0$, &c. This being premised, let us examine what becomes of the values which we found for the indeterminate quantities in equation (Q).

CCLXXXI. We had, 1st $A = \frac{1}{a}$, this value remains. 2nd, $B = -\frac{b}{a^3}$; but $b = 0$, therefore $B = 0$. 3rd, $C = \frac{2\,b^2 - a\,c}{a^5}$; but $b = 0$, therefore, $C = -\frac{a\,c}{a^5} = -\frac{c}{a^4}$. Proceeding in this manner, we obtain $D = 0$; $E = \frac{3\,a^2\,c^2 - a^3\,e}{a^9} = \frac{3\,c^2 - a\,e}{a^7}$, $F = 0$; $G = \frac{8\,a\,c\,e - 12\,c^3 - a^2\,g}{a^{10}}$ &c. But comparing the proposed equation (S) with that of (P), we get $a = 1$, $c = \frac{1}{2\,.\,3}$, $e = \frac{3}{2\,.\,4\,.\,5}$, $g = \frac{3\,.\,5}{2\,.\,4\,.\,6\,.\,7}$; substituting these values, we have those for the indeterminate quantities, $A = 1$, $C = -\frac{1}{2\,.\,3}$, $E = \frac{1}{2\,.\,3\,.\,4\,.\,5}$, $G = -\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}$; whence we can readily determine the value and sign of any number of terms.

—

In the present case $m =$ arc A, and $y = sin. A$; the equation (Q) then becomes:

$$(W)\ sin.\ A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7} + \&c.$$

By means of this series, which is infinite, we can obtain as many decimals as are required for the sine of any arc whatever, (see the values of arcs given at the foot of page vii.)

We are now prepared to enter into the

Theory of Logarithms.

CCLXXXII. In this theory the fundamental equation $b^\lambda = n$, in which n represents any number whatever, λ its logarithm, and b the base of the system. Whence are deduced the three following consequences.

CCLXXXIII. 1st. That each different value of the *base*, constitutes a different *system* of logarithms; the number of systems is therefore infinite. It may be superfluous to remark, that by changing b, it is necessary to change λ, on which the equality with the same number n depends. It is equally obvious that we could not attain the proposed end, by making $b = 1$.

2nd. That *the logarithm of unity is always zero.* For $b^\lambda = 1$, can only be obtained when $\lambda = 0$.

3rd. That *the logarithm of the base is always unity.* Indeed, when $b^\lambda = b$, we must necessarily have $\lambda = 1$.

CCLXXXIV. Let us observe that the logarithm must be some function of the number; and because the function becomes zero when $n = 1$, we make $n - 1 = x$, and its logarithm, a function of x; we may now make use of the method of indeterminate series, (art. 275...280).

CCLXXXV. Be the equation

$log.\ (1 + x) = M\ x + N\ x^2 + P\ x^3 + Qx^4 + \&c.$ Now because $log.\ (1 + x)^2 = log.\ (1 + 2\ x + x^2) = 2\ log.\ (1 + x)$, we may substitute in the series, $(2\ x + x^2)$ for x, $4\ x^2 + 4\ x^3 + x^4$ for x^2; $8\ x^3 + 12\ x^4 + \&c.$ for x^3; $16\ x^4$ &c. for x^4, and so on: the series, thus transformed, is doubled. Let the equation be arranged in the following order.

$$2\ log.\ (1+x) = \begin{cases} 2\,M\,x + \quad M\,x^2 + 4\,N\,x^3 + \quad N\,x^4 + \&c. \\ \qquad\qquad + 4\,N\,x^2 + 8\,P\,x^3 + 12\,P\,x^4 + \&c. \\ \qquad\qquad\qquad\qquad\qquad + 16\,Q\,x^4 + \&c. \end{cases}$$

we have also $-2\ log.\ (1+x) =$

$$-2\,M\,x - 2\,N\,x^2 - 2\,P\,x^3 - 2\,Q\,x^4 - \&c.$$

equalising the sum of the coefficients of each column, separately added, to zero, according to the principles in (art. 279), we find: $2\,M - 2\,M = 0$; M therefore remains indeterminate, then $M + 4\,N - 2\,N = 0$, whence $N = -\frac{M}{2}$. Substituting this value of N in the 3rd column, we get $P = \frac{M}{3}$. From the 4th column, we obtain $Q = -\frac{M}{4}$. The 5th column, by continuing the operation, will give $R = \frac{M}{5}$, and so forth. These values substituted in the 1st equation of art. (285), produce the following equation, which is the solution of the problem

$$(A)\quad log.\ (1+x) = M\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.\right)$$

CCLXXXVI. This confirms what is said in art. (283), of the infinite number of systems of logarithms, it was necessary that the value of one of the indeterminates should remain arbitrary, and that it should have an immediate relation to the value of the base, (see art. 292.) This indeterminate quantity M, is called the *modulus*.

The system in which $M = 1$ is that of Napier. Logarithms calculated according to this system are called *natural or hyperbolic*; by this last name they continue, though very improperly, to be called.

CCLXXXVII. It is not very obvious how the equation (A) can give the logarithm of any number whatever; the series is indeed convergent in making $x \angle 1$ or $x = 1$, but if this value be increased by any quantity however small, the series becomes divergent. This inconvenience is remedied by using transformations, or by the method in (art. 310), which renders it convergent in all cases.

CCLXXXVIII. Making x negative, the equation (A) becomes

$$(B) \ldots log.(1-x) = M\left(-\frac{x}{1}-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\&c.\right)$$

this series shews, that if $x \angle 1$, the logarithm of a fraction is negative.

If again $x = 1$, we have $log.\ 0 = -\infty$. This results from the fundamental equation (282); for we can only have $b^\lambda = 0$, when $\lambda = -\infty$. We then have $b^{-\infty} = \frac{1}{b^\infty} = 0$; which we obtain from the trigonometrical equation $\frac{sin.\ A}{tan.\ A} = cos.\ A$, when $A = 90°$.

Lastly, if $x > 1$, whatever be the value of x, the sum of the series remains $-\infty$. Thus it appears that the logarithm of every negative number is equal to the infinite negative. The nature of these logarithms has excited long discussions amongst the most eminent mathematicians; yet the utility of this question is very questionable, for referring to the fundamental equation $b^\lambda = n$, which gives in fact no system of logarithms from negative numbers, as n can never be negative unless b be so. And by adopting for the base a negative value (which indeed would be useless) there would result for n, values alternately negative and positive, according as λ were an even or an odd number.

CCLXXXIX. Subtracting equation (B) from equation (A), and recollecting that by the ordinary rules of the logarithmic calculus, $log.\ (1+x) - log.(1-x) = log.\ \frac{1+x}{1-x}$, we have

$$(C) \ldots log.\left(\frac{1+x}{1-x}\right) = 2\,M\left(\frac{x}{1}+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}+\&c.\right)$$

CCXC. Let it be required for example, to find the hyperbolic logarithm of 2; in resolving the equation $\frac{1+x}{1-x} = 2$, we find $x = \frac{1}{3}$, and because $M = 1$, (286,) we get

$$log.\ 2 = 2\left(\frac{1}{3}+\frac{1}{3\,.\,3^3}+\frac{1}{5\,.\,3^5}+\frac{1}{7\,.\,3^7}+\frac{1}{9\,.\,3^9}+\&c.\right)$$

It is evident that this series is much more convergent than that of equation (A). The calculation is as follows:—

$$\frac{1}{3} = 0{,}33333333$$

$$\frac{1}{3 \, . \, 3^3} = \;\;,01234568$$

$$\frac{1}{5 \, . \, 3^5} = \;\;,00082304$$

$$\frac{1}{7 \, . \, 3^7} = \;\;,00006532$$

$$\frac{1}{9 \, . \, 3^9} = \;\;,00000565$$

$$\frac{1}{11 \, . \, 3^{11}} = \;\;,00000051$$

$$\frac{1}{13 \, . \, 3^{13}} = \;\;,00000005$$

$$0{,}34657358$$

which multiplied by 2

gives the hyperbolic log. of $2 =$ 0,69314716 correct to the 7th figure; in order to have the last figure correct, we must take one or two figures more than required.

CCXCI. Calling (L) the hyperbolic logarithm of any number, l, the logarithm corresponding to the same number in another system, and S the sum of any one of the three series A, B or C; we have $L = S$. and $l = M\,S = M\,L \; \therefore \; L = \frac{l}{M}$.

Hence, *multiplying the hyperbolic logarithm of a number by the modulus of another system, we have the logarithm corresponding to the same number in another system, and dividing the logarithm of any system whatever, by the modulus of that system, we obtain the corresponding hyperbolic logarithm.*

CCXCII. Let us now make $\frac{1+x}{1-x} = b$, whence $x = \frac{b-1}{b+1}$.

By substituting this value in equation (C) and dividing by M, we have (283)

$$\frac{log.\, b}{M} = \frac{1}{M} = 2\left[\frac{b-1}{b+1} + \frac{1}{3}\left(\frac{b-1}{b+1}\right)^3 + \frac{1}{5}\left(\frac{b-1}{b+1}\right)^5 + \frac{1}{7}\left(\frac{b-1}{b+1}\right)^7 + \frac{1}{9}\left(\frac{b-1}{b+1}\right)^9 + \&c.\right]$$

which series shews how the value of the modulus depends on the base. One only of the two values can be a whole number.

CCXCIII. The increments of logarithms are easily found by the preceding formulæ. If any number n, receives an increase, which we shall indicate by ∂n, what will be the corresponding augmentation of log. n? this value is found immediately by the equation log. $n + \partial$ log. $n =$ log. $(n + \partial n)$, whence we conclude that, ∂ log. $n =$ log. $(n + \partial n) -$ log. $n =$ log. $\frac{n + \partial n}{n}$ $=$ log. $\left(1 + \frac{\partial n}{n}\right)$.

CCXCIV. If this last expression be reduced in series by means of the formula A (285), in making $x = \frac{\partial n}{n}$, we get

$$(D)\ldots\ \partial\, log.n = M\left[\frac{\partial n}{n} - \tfrac{1}{2}\left(\frac{\partial n}{n}\right)^2 + \tfrac{1}{3}\left(\frac{\partial n}{n}\right)^3 - \tfrac{1}{4}\left(\frac{\partial n}{n}\right)^4 + \&c.\right]$$

And this is the differential formula, of which only the first term is taken when the differences are very small.

If n, instead of increasing, diminishes, the negative sign is perfixed to ∂n. The series (D) then, is entirely negative; whence it evidently follows, that as the number diminishes the logarithm diminishes also.

CCXCV. The next formula (F) is however much more convergent, and may with advantage be substituted to all the preceding.

Taking $\frac{1+x}{1-x} = \frac{n + \partial n}{n}$, we get $x = \frac{\partial n}{2n + \partial n}$. Substituting these values in equation (C), and writing ∂ log. n instead of log. $\frac{n + \partial n}{n}$, (293) we have

$$(F)\ldots\ \partial\, log.\, n = 2\, M\left[\frac{\partial n}{2n + \partial n} + \frac{1}{3}\left(\frac{\partial n}{2n + \partial n}\right)^3 + \frac{1}{5}\left(\frac{\partial n}{2n + \partial n}\right)^5 + \&c.\right]$$

This series, which is beyond comparison more convergent than the series (D), is used to calculate the difference between a known logarithm and one greater or less. In the last case, it is sufficient to give to ɜ n the negative sign.

In making the log. $n =$ log. 1 ; the series (F) serves to calculate the logarithm of any positive number, either greater, or less than unity.

CCXCVI. Having found, art. (289), an expeditious series by which to calculate the hyperbolic logarithm of 2, (the formula (F) gives the same series when $n = 1$ and ɜ $n = 1$,) we can obtain the log. of 4, because log. $4 =$ log. $2^2 = 2$ log. 2. Having the logarithm of 4, we easily obtain, by means of the formula (F), the log. of 5, a work of many days labor of the most tedious kind to the first calculators of logarithms, to whom the preceding formulæ were unknown. In this case $n = 4$, ɜ $n = 1$, consequently the equation (F) becomes

$$ɜ\ \mathit{log.}\ 4 = 2\left(\frac{1}{9} + \frac{1}{3\ .\ 9^3} + \frac{1}{5\ .\ 9^5} + \frac{1}{7\ .\ 9^7} + \&c.\right)$$

Four terms of this series are sufficient to obtain the quantity we have to add to the logarithm of 4, in order to get the hyperbolical logarithm of 5 correct to seven places of decimals.

The calculation is as follows:—

$$\begin{array}{rcl}
\frac{1}{9} & = & 0{,}111111111 \\[4pt]
\frac{1}{3\ .\ 9^3} & = & {,}000457247 \\[4pt]
\frac{1}{5\ .\ 9^5} & = & {,}000003387 \\[4pt]
\frac{1}{7\ .\ 9^7} & = & {,}000000030 \\
\hline
\mathit{Sum} & = & {,}111571775 \\
\hline
\mathit{multiplied\ by}\ 2 & = & 0{,}22314355 \\
\mathit{hyperbolic\ log.\ of}\ 4 & = & 1{,}38629436 \\
\hline
\mathit{hyper.\ log.\ of}\ 5 & = & 1{,}60943791
\end{array}$$

we have employed here 9 figures to be sure of the correctness of 7. Adding this log. of 5, to that of 2 we get log. $10 =$ 2,30258509 +.

CCXCVII. If instead of log. of 4, we had employed the log. of 6, to obtain that of 5, then the formula (F) would have given $-\dfrac{\delta n}{2n - \delta n} = -\dfrac{1}{11}$; whence it results, that for an equal value of δn, the series is more convergent when we have to descend from a greater to a smaller logarithm. The following is the calculation

$$\delta\ log.\ 6 = -2\left(\frac{1}{11} + \frac{1}{3\ .\ 11^3} + \frac{1}{5\ .\ 11^5} + \frac{1}{7\ .\ 11^7} + \&c.\right)$$

$$\frac{1}{11} = 0{,}090909091$$

$$\frac{1}{3\ .\ 11^3} = 0{,}000250438$$

$$\frac{1}{5\ .\ 11^5} = 0{,}000001242$$

$$\frac{1}{7\ .\ 11^7} = 0{,}000000007$$

$$\textit{Sum} = 0{,}091160778$$

$$\textit{multiplying by } 2 = 0{,}18232156$$
$$\textit{hyperbolic log. of } 6 = 1{,}79175947$$

$$\textit{hyperbolic log. of } 5 = 1{,}60943791$$

CCXCVIII. The second member of the equation (D) shews the *first* difference between any logarithm and a greater; and with all the terms negative, (294), the difference between any logarithm and a less. But the sum of any number of terms having the same sign, is greater than the sum of the same terms with alternate signs. Hence, for any given numerical increase, the lesser logarithm increases faster than the greater.

CCXCIX. Of any three numbers in arithmetical progression the *second* difference of logarithms must be negative*.

				1st diff.	2nd diff.
* Numbers......	342	Log.	5340261	+ 12680	− 47
	343		5352941	+ 12633	
	344		5365584		

This difference consists of twice the even terms of the series (D). In fact the differences are found by the subtraction of the former quantity or first in order, from the subsequent or next in order, that is, the lesser logarithm from the greater; since the difference between terms all negative, must have signs all positive; and this difference, which is the greatest (298) is again subtracted from the less which has alternate signs; hence, the second difference becoming negative, and the odd terms vanish. This will be easily understood by writing the first differences between three successive logarithms, than the difference of these differences, and the present article will become clear.

CCC. Let the logarithm of $(n - \delta n)$ and that of n be given, that of $(n + \delta n)$ is found by subtracting the second difference $\left[\frac{M}{1}\left(\frac{\delta n}{n}\right)^2 + \frac{M}{2}\left(\frac{\delta n}{n}\right)^4 + \frac{M}{3}\left(\frac{\delta n}{n}\right)^6 + \&c\right]$, from the known difference between log. $(n - \delta n)$ and log. n, and adding the remainder to log. n.

By the two last series, we find that *as the number, whose corresponding logarithm is to be deduced from that of the next greater or less, increases, the series becomes more divergent or otherwise, and consequently fewer or more terms will be required.*

CCCI. We are now prepared to determine with ease the modulus of the *tabular* or *vulgar* logarithm, viz. that of the ordinary tables. As in this system, log. $10 = 1$, if in the equation $l = M\,L$ (291), we suppose $l = 1$, L shall be the hyperbolic logarithm of 10 (296); which continued to 28 decimal places,

$L = 2{,}3025850929940456840179914547 = \frac{1}{M}$, which equation

gives $M = 0{,}4342944819032518276511289189$. Multiplying this number, by a hyperbolic logarithm, we obtain the corresponding vulgar logarithm (291); and by multiplying the preceding number, viz. $\frac{1}{M}$, by a vulgar logarithm, the product is the corresponding hyperbolic logarithm, (291.)

CCCII. The number 10 is the base of the ordinary system, or the system of Briggs, who was the first who calculated the ordinary tables. This system is the most convenient, since by a simple change in the characteristic or index, the log. of any number is also the log. of the same number multiplied or divided

by any power of 10. For example, (315) the log. of 2 is also the log. of 200, of 20000, of 0,002, &c. provided that the characteristic be increased or diminished accordingly. The hyperbolic logarithms have no such advantage.

CCCIII. *To find the number corresponding to a given logarithm.* This problem is easily resolved by dividing by M, the equation (A) and converting it by the method of returning a series (275). Let $(1 + x)$ be the number sought, and by way of abridging, let $\frac{log.\,(1+x)}{M} = y$; the equation ($A$) converted gives* $x = y + \frac{y^2}{2} + \frac{y^3}{2\,.\,3} + \frac{y^4}{2\,.\,3\,.\,4} + \text{\&c.}$ consequently we obtain by adding one, the number sought

$$1 + x = 1 + y + \frac{y^2}{2} + \frac{y^3}{2\,.\,3} + \text{\&c.}$$

In general, for any number whatever n, we have

$$G \quad . \quad . \quad n = 1 + \left(\frac{log.\,n}{M}\right)^1 + \frac{1}{2}\left(\frac{log.\,n}{M}\right)^2 + \frac{1}{2\,.\,3}\left(\frac{log.\,n}{M}\right)^3 + \frac{1}{2\,.\,3\,.\,4}\left(\frac{log.\,n}{M}\right)^4 + \text{\&c.}$$

CCCIV. If in this series we suppose $\log n = 1$, and $M = 1$ the value of n will be (283, 286), the base of the hyperbolic logarithm: consequently

$n = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$ and which

---

* The conversion of the series will be as follows:

$$\begin{array}{llllllll}
\text{Equation } (A) & y = x & - \frac{x^2}{2} & + \frac{x^3}{3} & - \frac{x^4}{4} & + \;.\;.\;.\;. \\
\hline
\text{consequently} & x = & A\,y + & B\,y^2 + & C\,y^3 + & D\,y^4 + \;.\;.\;.\;. \\
 & -\frac{x^2}{2} = & & -\frac{A^2}{2}y^2 - & A\,B\,y^3 - & A\,C\,y^4 - \;.\;.\;.\;. \\
 & & & & & -\frac{B^2}{2}y^4 \\
 & +\frac{x^3}{3} = & & & \frac{A^3}{3}y^3 + & A^2\,B\,y^4 \\
 & -\frac{x^4}{4} = & & & & -\frac{A^4}{4}y^4 \\
 & -y & & & &
\end{array}$$

by reduction gives $n = 2{,}718281828459045235360 28^*$. This number is of great use in the integral calculus, where it is usually indicated by the letter e.

CCCV. Comparing the formula (D) with the general formula (P) (275), we have $\frac{s\cdot\log.n}{M} = m$; $s\cdot n = y$; $\frac{1}{n} = a$; $-\frac{1}{2n^2} = b$, and so on. With these values of a, b, c, &c. we

---

$Ay - y = 0 \quad \therefore \quad A = 1$; $B - \frac{1}{2} = 0 \quad \therefore \quad B = \frac{1}{2}$; $C - \frac{1}{2} + \frac{1}{3} = 0 \quad \therefore \quad C = \frac{1}{6} = \frac{1}{2\,.\,3}$; $D - \frac{1}{6} - \frac{1}{8} + \frac{1}{2} - \frac{1}{4} = 0$ $\therefore \quad D = \frac{1}{24} = \frac{1}{2\,.\,3\,.\,4}$.

* The two first terms 2,0000000000

$\frac{1}{2}$ 0,5000000000

$\frac{1}{2\,.\,3}$ 0,1666666666

$\frac{1}{2\,.\,3\,.\,4}$ 0,0416666666

$\frac{1}{2\,.\,3\,.\,4\,.\,5}$ 0,0083333333

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6}$ 0,0013888888

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}$ 1984127

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8}$ 248016

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9}$ 27557

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10}$ 2755

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10\,.\,11}$ 251

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10\,.\,11\,.\,12}$ . . . 21

$\frac{1}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10\,.\,11\,.\,12\,.\,13}$ 2

hyperbolic Base = 2,718281828

obtain those of A, B, C, &c. in the formula (Q) (276); and the series (D) converted, becomes

$$(K) \quad \ldots\ldots\ldots \quad \Delta n = n\left[\frac{\Delta\, log.n}{M} + \frac{1}{2}\left(\frac{\Delta\, log.n}{M}\right)^2 + \frac{1}{2.3}\left(\frac{\Delta\, log.n}{M}\right)^3 + \&c.\right]$$

CCCVI. This series shews the increase of a number, relatively to that of its logarithms. If the decrease of a logarithm were given, we should deduce from it the decrease of the number, by giving the negative sign to $\Delta\, log.\, n$ (294); the odd terms only in this case being negative.

CCCVII. The construction of the formula being now accomplished, it will be useful to prepare the factors M and $\frac{1}{M}$.

Taking the values from (301) we have

M =	0,43429	44819	03251	82765	11289
$2\,M$ =	0,86858	89638	06503	65530	22578
$3\,M$ =	1,30288	34457	09755	48295	33868
$4\,M$ =	1,73717	79276	13007	31060	45157
$5\,M$ =	2,17147	24095	16259	13825	56446
$6\,M$ =	2,60576	68914	19510	96590	67735
$7\,M$ =	3,04006	13733	22762	79355	79024
$8\,M$ =	3,47435	58552	26014	62120	90314
$9\,M$ =	3,90865	03371	29266	44886	01603

Again

$\frac{1}{M}$ =	2,30258	50929	94045	68401	79915
$\frac{2}{M}$ =	4,60517	01859	88091	36803	59829
$\frac{3}{M}$ =	6,90775	52789	82137	05205	39744
$\frac{4}{M}$ =	9,21034	03719	76182	73607	19658
$\frac{5}{M}$ =	11,51292	54649	70228	42008	99573
$\frac{6}{M}$ =	13,81551	05579	64274	10410	79487
$\frac{7}{M}$ =	16,11809	56509	58319	78812	59402

$$\frac{8}{M} = 18{,}42068 \quad 07439 \quad 52365 \quad 47214 \quad 39316$$

$$\frac{9}{M} = 20{,}72326 \quad 58369 \quad 46411 \quad 15616 \quad 19231$$

CCCVIII. By means of these preparations, the conversion of the vulgar into hyperbolic logarithms and reciprocally, is reduced to simple additions. Let it be required, for example, to find the hyperbolic *log.* of 10,09 ; take the corresponding vulgar *log.* (with eight decimals, if convenient, in order to have the seventh exact) this *log.* is 1,00389117, disposing of the values of $\frac{1}{M}$ for each cypher which composes the *log.*, we get

$$\begin{array}{lr}
\frac{1}{M} = & 2{,}30258509 \\
\frac{0{,}003}{M} = & 690776 \\
\frac{0{,}0008}{M} = & 184207 \\
\&c. & 20723 \\
 & 230 \\
 & 23 \\
 & 16 \\
\hline
\text{sum} = & 2{,}3115448
\end{array} = \text{hpy. } log. \text{ of } 10{,}09.$$

CCCIX. This operation is so short and simple, that it will frequently be preferred in the formation of logarithms. This example, it may be observed, shews how convergent the formula (F) (295) is, as it is only necessary to calculate the first term of the series to get the quantity to be added to 2,3105532 the *log.* of 10,08, exact to 7 or 8 decimal places, to obtain the *log.* of 10,09. In this case $n = 10{,}08$, and $\Delta n = 0{,}01$; the first term of the series gives

$$\Delta\, log.\, n = 2 \times \frac{0{,}01}{20{,}17} = 0{,}0009916$$

$$log. \text{ of } 10{,}08 \quad . \quad . \quad = 2{,}3105532$$

$$\text{sum of } log. \text{ of } \quad 10{,}09 = 2{,}3115448$$

Let it be remembered that in every calculation of approximation, when it is necessary to employ a known quantity to obtain

another, this to obtain a third and so on, we must take the calculation with one, sometimes two, three &c. decimals, as the case may be, more than are actually required, in order to have these with exactness, otherwise the error of the decimals might be of the same kind in each quantity, and thus accumulating itself, become sensible.

CCCX. The *log.* of 10 being known, (296) there remains no longer any difficulty about the divergency of the series (A) (287), as it is easy to render it convergent for the greatest number. Let it be required, for example, to find the hyperbolic *log.* of 12389. Seeing that $12389 = 1,2389 \times 10000 = 1,2389 \times 10^4$. Therefore *log.* of $12389 = 4$ *log.*$10 +$ *log.* $1,2389$. Making $x = 0,2389$ the series (A) becomes convergent, and gives a logarithm which added to 4 *log.* 10 will be the logarithm sought.

CCCXI. If instead of 12389, the number was 0,12389, we should say *log.* of $0,12389 =$ *log.* of $\frac{1,2389}{10} =$ *log.* $1,2389$ — *log.* 10. In the same manner for 0,0012389, we can substitute $\frac{1,2389}{1000}$; and thus in all similar cases. It must however be remembered, that here the greater logarithm being negative, we subtract the less from the greater, and prefix to the remainder the sign —, according to the ordinary rules. For we cannot, for the reasons assigned in art. (302,) apply to the hyperbolical *log.* of fractions the expedient adopted (318) for the vulgar logarithm.

CCCXII. The series (F), is however much preferable to the series (A), in calculating the logarithm of a given number. Indeed, reducing the number 12389 to the form 1,2389, and making $n = 1$, we have $z - n = 0,2389$. With these values we need only calculate the three first terms of the series (F) to obtain the *log.* sought exact to seven decimal places, whilst we should be obliged to calculate no less than eight terms of the series (A) to arrive at the same result.

CCCXIII. The more the first figure of the given number surpasses unity, the less the series (F) under the form (295) becomes convergent. Let it be required, for example, to obtain the *log.* of 3412. Writing 3412×1000, and making $n = 1$,

3 H

we get $\vartheta n = 2,412$, consequently $\dfrac{\vartheta n}{2n + \vartheta n} = \dfrac{2,412}{4,412}$. This fraction is greater than 0,5. Employing the same series but making ϑn negative (295) we are enabled to descend to a lesser logarithm; for this purpose we take, $n = 1$ as before, and $3412 = 0,3412 \times 10000$. Hence, seeing *log.* $1 = 0$, we proceed to *log.* 0,3412, and we have $\vartheta n = -0,6588$. Consequently $\dfrac{\vartheta n}{2n + \vartheta n} = -\dfrac{0.6588}{1,3412}$; which fraction is less than 0,5 as was desired. Theuce 0,5 is the limit of the values $\dfrac{\pm \vartheta n}{2n \pm \vartheta n}$, to determine which of the two forms of the series (F) ought to be employed.

CCCXIV. If from a known logarithm it is required to obtain another by means of the series (F) as is done (296 and 309) that series will be found the more convergent as the number n, whose log. is known, is greater and as ϑn smaller.

Use of the Logarithmic Tables.

CCCXV. Let c be the characteristic or index of a logarithm, then $c + 1$ shews how many figures, of the number corresponding to that logarithm, are integers. For example 0,8665236, is the logarithm of 7,354, and 2,8665236 is the *log.* of 735,4; in the first case therefore $c = 0$; thence the integral part of the number can have only one single figure 7, a whole number and 354 are decimals. In the second case $c = 2$; the corresponding number 735,4, has the three first figures integrals. Agreeably to this, the characteristic of logarithms, corresponding to decimal fractions, must be less than zero, that is, it must be negative.

The logarithm of fractions, vulgar or decimal, can be expressed in three different ways. Let us take for example, the fraction $\frac{2}{3}$. According to the theory of logarithm *log.* $\frac{2}{3} =$ *log.* 2 $-$ *log.* $3 = 0,30103 - 0,47712$, (employing for shortness' sake, but five decimals;) in effecting the subtraction, we have *log.* $\frac{2}{3} = -0,17609$. But this method, though exact, is wholly proscribed, on account of the inconvenience of using negative

logarithms. The chief reason is, that the number corresponding to such a logarithm can not be had by the tables, unless we regard the number as the denominator of a fraction having unity for its numerator. Hence it would follow that the number corresponding to everyd ifferent negative logarithm must have a different denominator, and that consequently we get fractions which do not possess the advantage of decimal fractions, viz., to be immediately comparable between themselves.

CCCXVI. The second method adopted by some authors consists in effecting the subtraction of the characteristic alone. For example $log. \frac{3}{4}$ or $log. \frac{75}{100} = 1,87506 - 2,00000 = -\bar{1} + 0,87506$, which they express thus: $\bar{1}, 87506$; and in the same manner $log. \frac{75}{1000}$ or $log. 0,075 = \bar{2},87506$; $log. 0,0075 = \bar{3},87596$; $log. 0,00075 = \bar{4},87506$, &c. This method is expeditious, but it is not much in use, because it also necessitates the employment of the negative sign, and that it requires some attention to avoid confounding the two operations, the addition of the decimals, and the subtraction of the characteristics. The following is an example of this method; having found the log of $\frac{75}{100}$, suppose it be required to find, by means of the log. tables, the product of 32 by $\frac{75}{100}$;

$$log.\ 32 = 1,50515$$

$$log.\ \frac{75}{100} = \bar{1},87506$$

$$\text{sum of } log.\ 32 + log.\ \frac{75}{100} = log.\ 24 = 1,38021$$

and such is indeed the $log.$ of 24.

CCCXVII. Proceeding to the 3rd method, which seems to be the one most generally adopted. As we cannot, in any ordinary calculation at least, commit a mistake of ten thousand millions, we run no risk in supposing the characteristic increased by ten whenever it is found necessary, in order to have it always positive. Thus in the proposed example, making $log. \frac{75}{100} = 11,87506 - 2,00000$, we have $log. 0,75 = 9,87506$; but ac-

cording to the general rule (315),9,87506 = *log.* of 7500000.000; that the characteristics then thus confounded should lead into error, it would be necessary to make the mistake employing 7500 millions instead of 0,75 or $\frac{3}{4}$ of unity, which in all ordinary calculations is impossible. We say ordinary because in calculations connected with geodesic operations, such as are at the present time carrying on in India, under Colonel Everest, and in which the utmost nicety is used, the quantities brought into calculation affect frequently the eighth decimal only, so as to augment or lessen by unity, the last or 7th decimal in the tabular logarithm in use; in similar calculations, preference is sometimes given to negative indices.

CCCXVIII. Observing that 9,87506 = 10 — 0,12494 we conclude, 1st, that it is easy to convert every negative logarithm into a positive one (315); 2nd, that the characteristic of the log. of a fraction is always less than 10.

Adding 10 to the characteristic, we have *log.* 0,75 = 9,87506. For the same reason *log.* 0.075 = 8,87506; *log.* 0,0075 = 7,87506; *log.* 0,00075 = 6,87506, &c. Hence the rule that: *the logarithm of a decimal fraction, has for its characteristic the complement to 9 of the number of cyphers commencing the given decimal fraction.*

From this rule we learn, that when we have the logarithm of a decimal fraction, we must look in the tables for that number to which this logarithm answers, without taking the characteristic into account, then we prefix zero to the number so found, followed by a decimal and as many cyphers after it as there are wanting units in the characteristic to make up the complement of 9. All this will be more readily understood in paying attention to the preceding and following examples.

CCCXIX. Recollecting that the log. of a decimal fraction is always less than 10; it follows, *that, in the addition of the logarithm of decimal fractions, we must not take the tens into account in the characteristic of the sum.* They are then suppressed, consequently if the log. of a sum is to represent that of a whole number, the characteristic is exact and without increase; but if this logarithm is to be that of a fraction, the characteristic is to be taken as by the rule (318).

Let it be required for example to find by the logarithm tables, the product of 36 by $\frac{3}{4}$

we have *log*. of 36 = 1,55630
log. of . 75 = 9,87506

sum = 11,43136

Neglecting the tenths of the sum, we have only 1,43166, the error then resulting from the rule in art. (315) vanishes, as this log. is the exact number of 36 by . 75 = 27.

But if we want the product of 0,75 by 0,36 we have

log. of . 36 = . . 9,55630
log. of . 0,75 = . . 9,87506

19,43136

By the suppression of the tenths, the sum is reduced to 9,43136; and it is easy to see, according to the rule given in art. (318), that the characteristic is here that of a decimal fraction, and that the logarithm is that of 0,27 = 0,36 × 0,75.

CCCXX. As a general rule, the tenths of the characteristic are always to be suppressed. Thus, for example, let it be required to elevate a decimal fraction to a given power, the square of 0,4 being 0,16, we have 2 *log*. 0,4 = 19,20412 we write only 9,20412. For the same reason, if we wish to have by means of the log. tables, the cube of 0,4 = 0,064 we have 3 *log*. 0,4 = 28,80618 and we write only 8,80618. We see that in the characteristic a tenth has been suppressed for the 2nd power, and two tenths for the 3rd, we must then suppress for the 4th power three tenths, and so on.

Whence it follows, that in the extraction of roots, which is the reverse operation, we must supply the neglected tenths, without which the calculation would not be exact. We must supply then, mentally at least, $m - 1$ tenths before the characteristic, m being the exponent of the radical.

For example, to have the 4th root of 0,0256 of which the logarithm is (318) 8,40824, before dividing this logarithm by 4, we must supply the characteristic with $m - 1$ tenths, in the present instance with three, or rather we suppose it to be written 38,40824, the quotient then is 9,60206 which is the *log*. of 0,4 = $\sqrt[4]{0,0256}$.

CCCXXI. The logarithmic calculation becomes still more expeditious by the use of the *arithmetical complement* an operation by which *subtraction* is performed by means of addition. This process, and its principle, will be readily comprehended by an example. Let it be required to subtract 437 from 795.

The common process would be as follows:

From . . .	795
Subtract . . .	437
Remainder	358

The *arithmetical complement* of 437, or the number by which it falls short of 1000, is 563. Now, if this number be added to 795, and the first figure on the left be struck out of the sum, the result will be as in the former case.

To . . .	795
Add . . .	563
Sum	1358
Remainder sought	358

The principle on which this process is founded is easily explained. In the latter process we have first added 563, and then subtracted 1000. On the whole, therefore, we have subtracted 437, since the number actually subtracted exceeds the number previously added by that amount.

Since therefore subtraction may be effected in this manner by means of addition, it follows that the calculation of all series, so far as an order of differences can be found in them which continues constant, may be conducted by the process of addition alone.

All this at first sight, appears to embarrass the calculation rather than to facilitate it. But practice will soon convince us of the advantage of employing the arithmetical complement. For example let it be required to find the product of 0,32 by 569,1 divided by 28.

The common process would be as follows:

to *log.* of 569,1 = . .	2,75519
add *log.* of 0,32 = . .	9,50515
sum of *logs.* . .	2,26034
subtract *log.* of 28 . . =	1,44716
Remains *log.* of 6,504 =	0,81318

But in making use of the arithmetical complement, the process will stand as follows:

log. of 569,1 = . .	2,75519
log. of 0,32 = . .	9,50515
arith. comp. *log.* of 28, = . .	8,55284
sum as before = . .	0,81318

This process is not only shorter, but it is considered more elegant, suffice it to say that we soon acquire the habit of writing the arithmetical complement of a logarithm with equal facility as the logarithm itself.

Before leaving the subject we may observe, that the tangents as far as 45° and all the sines (except the sine of 90°) are decimal fractions when we adopt the general supposition of $R = 1$. The characteristics of their logarithms are also in the tables agreeably to the rule given in art. (318). In some tables the tangents of arcs greater than 45°, have the characteristic 10, 11, &c. In these tables, the trigonometrical lines are not considered as parts of the $R = 1$, but as parts of $R = 10000000000$. In using these tables we should always neglect this tenth, making the more commodious and consistent supposition of $R = 1$.

CCCXXII. We now proceed to give some examples of the use, which may be made of logarithms in finding the numerical value of formulæ. It follows from what is said in art. 272, and from the definition of logarithms, by which we are furnished with the equation $b^{\lambda} = n$, that

$$l\,(A\,B) = l\,A + l\,B, \qquad l\left(\frac{A}{B}\right) = l\,A - l\,B,$$

$$l\,A^m = m\,l\,A, \qquad l\,A^{\frac{1}{n}} = \frac{1}{n}\,l\,A.$$

Applying these principles to the formula

$$\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3.\,E.\,F}},$$

which is very complicated, we find

$$l\,(A^2\sqrt{B^2 - C^2}) = l\,[A^2\sqrt{(B + C)\,(B - C)}] =$$
$$2\,l\,A + \tfrac{1}{2}\,l\,(B + C) + \tfrac{1}{2}\,l\,(B - C),$$

and $\quad l\,(C\sqrt[5]{D^3.E.F}) = l\,C + \tfrac{3}{5}\,l\,D + \tfrac{1}{5}\,l\,E + \tfrac{1}{5}\,l\,F,$

and, consequently,

$$l\left(\frac{A^2\sqrt{B^2 - C^2}}{C\sqrt[5]{D^3.\,E.\,F}}\right) =$$

$$2\,l\,A + \tfrac{1}{2}\,l\,(B + C) + \tfrac{1}{2}\,l\,(B - C) - l\,C - \tfrac{3}{5}\,l\,D - \tfrac{1}{5}\,l\,E - \tfrac{1}{5}\,l\,F.$$

If we take the arithmetical complements of $l\,C$, $\tfrac{3}{5}\,l\,D$, $\tfrac{1}{5}\,l\,E$, $\tfrac{1}{5}\,l\,F$, designating them by C', D', E', F', instead of the preceding result, we have

$$2\,l\,A + \tfrac{1}{2}\,l\,(B + C) + \tfrac{1}{2}\,l\,(B - C) + C' + D' + E' + F',$$

only we must observe to subtract from the sum as many units of the same order with the complements, as there are complements taken, that is 4. When we have found the logarithm of the proposed formula, the tables will show the number to which this logarithm belongs, which will be the value sought.

The calculation of the formula for the radius of projection

$$r = a \cot. l \left(\frac{1+n}{1+n \cos. 2\, l}\right)^{\frac{1}{2}}$$

in which a the equatorial radius $= 3486852{,}4$ fathoms;
b the polar radius $= 3475419{,}66$
and l the latitude of 20°
being rather complicated, we give the

Type of the Calculation.

$$n = \frac{e^2}{2 - e^2};\ e^2 = \frac{a^2 - b^2}{a^2} = \text{(by making } a = 1) \quad 1 - \beta^2$$

$log. b = log. 3475419{,}66 =$	. .	6,5410073		
$log. a = log. 3486852{,}4 =$	. .	6,5424336		
$log. b$ in parts of equatl. radius $=$		9,9985737		
$log. \beta^2 =$	. . . .	9,9971474	=	0,9934532
Therefore $1 - \beta^2 = 0{,}0065468 = e^2$; and $2 - e^2$			=	1,9934532
$log. e^2 =$	. .	7,8160291		
$- log. (2 - e^2) =$	. .	0,2996061		
$log. n =$	. .	7,5164230	=	0,00328415
$log. \cos. 2\, l =$	. .	9,8842540		
$log. (n \cos. 2\, l) =$	. .	7,4006770	=	0,0025158
$log. (1 + n) =$	. .	0,0014239		
$log. (1 + n \cos. 2\, l) =$	. .	0,0010911		
$log. \dfrac{1+n}{1+n \cos. 2\, l} =$	. .	0,0003328		
$log. \left(\dfrac{1+n}{1+n \cos. 2\, l}\right)^{\frac{1}{2}} =$	. .	0,0001664		
$log. a =$	. .	6,5424336		
$log. \cot. l =$	. .	0,4389341		
$log. r =$	. .	6,9815341	=	9583720

fathoms.

CCCXXIII. Logarithms are of the most frequent use in finding the fourth term of a proportion. It is evident, that if $a : b :: c : d$ we have

$$d = \frac{b\,c}{a}, \text{ whence } l\,d = l\,b + l\,c - l\,a;$$

that is, *the logarithm of the fourth term sought, is equal to the sum of the logarithms of the two means, diminished by the logarithm of the known extreme,* or rather, *to the sum of the logarithms of the means, plus the arithmetical complement of the logarithm of the known extreme.*

CCCXXIV. If we take the logarithms of each member of the equation $\frac{b}{a} = \frac{d}{c}$, which presents the character of a proportion, we have

$$l\,b - l\,a = l\,d - l\,c;$$

whence it follows, that the four logarithms

$$l\,a \,.\, l\,b : l\,c \,.\, l\,d$$

form an equidifference (252)

The series of equations,

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d}, \text{ \&c. } (260)$$

leads also to

$$l\,b - l\,a = l\,c - l\,b = l\,d - l\,c = l\,e - l\,d, \text{ \&c.}$$

and hence we infer, that the progression by quotients,

$$\div\!\div\, a : b : c : d : e, \text{ \&c.}$$

corresponds to the progression by differences,

$$\div\, l\,a \,.\, l\,b \,.\, l\,c \,.\, l\,d \,.\, l\,e, \text{ \&c.}$$

and, consequently, *the logarithms of numbers in progression by quotients, form a progression by differences.*

CCCXXV. If we have the equation $b^x = c$, we may easily resolve it by means of logarithms; for as $l\,b^x$ is equal to $x\,l\,b$, we have $x\,l\,b = l\,c$, and, consequently, $x = \frac{l\,c}{l\,b}$. The equation $b^{c^x} = d$, may be resolved in the same manner; making $c^x = u$, we have

$$b^u = d,\; u\,l\,b = l\,d,\; u = \frac{l\,d}{l\,b}, \text{ or } c^x = \frac{l\,d}{l\,b};$$

again taking the logarithms, we find

$$z\,l\,c = l\left(\frac{l\,d}{l\,b}\right) = l\,l\,d - l\,l\,b \text{ and } z = \frac{l\,l\,d - l\,l\,b}{l\,c}.$$

In this last expression, $l\,l\,b$ represents the logarithm of the logarithm of b, and is found by considering this logarithm as a number. The quantities, b^z, b^{c^z}, and all which are derived from them, are called *exponential quantities.*

As it is very important to practise in the use of logarithm in algebraic calculations, we give some examples.

It happens sometimes that we find less advantage in taking the base $= 10$, than using a different system; in this case, it is easy with the aid of tables, such as those of Briggs, to calculate any other logarithm in the new system. For example, the *log.* of $\frac{2}{3}$, in the system of which the base is $\frac{5}{7}$, is $\frac{log.\ \frac{2}{3}}{log.\ \frac{5}{7}}$ $= \frac{log.\ 2 - log.\ 3}{log.\ 5 - log.\ 7}$, the base here is whatever we wish, and if we take it $= 10$, then all becomes known, and we have $\frac{-0{,}17609125}{-0{,}14612804} = 1{,}2050476$ for the log. sought.

Again $log.\ \frac{2}{3}$, in the system $\frac{3}{2}$, is $\frac{log.\ \frac{2}{3}}{log.\ \frac{3}{2}} = \frac{log.\ 2 - log.\ 3}{log.\ 3 - log.\ 2}$, or -1, which is evident, as the equation $n = b^\lambda$ becomes here $\frac{2}{3} = \left(\frac{3}{2}\right)^\lambda = \left(\frac{2}{3}\right)^{-\lambda}$, where λ is obviously -1.

1. $log.\ \frac{a^n\,b^m\,c^{-r}}{d^s\,e^{-p}} = n\ log.\ a + m\ log.\ b - r\ log.\ c - s\ log.\ d + p\ log.\ e.$

2. $log.\left(a^{\frac{m}{n}}\,b^{-\frac{r}{s}}\,c^{-\frac{1}{n}}\right) = \frac{m}{n}\ log.\ a - \frac{r}{s}\ log.\ b - \frac{1}{n}\ log.\ c.$

3. $log.\ \sqrt[m]{a^{-m}\ b^{-\frac{n}{m}}\ c^{\frac{p}{q}}} = -\ log.\ a - \frac{n}{m^2}\ log.\ b + \frac{p}{mq}\ log.\ c.$

4. $\frac{a\sqrt[m]{b^{\frac{1}{n}}\ d^{\frac{m}{n}}\ e^{-\frac{p}{q}}}}{h\sqrt[-r]{f g^{\frac{2}{r}}}} = log.\ a + \frac{1}{mn}\ log.\ b + \frac{1}{n}\ log.\ d - \frac{p}{mq}\ log.\ e + r\ log.\ h - \frac{1}{2}\ log.\ f + \frac{1}{r}\ log.\ g.$

5. $log.\ (a^3 \sqrt[4]{a^3}) = 3\ log.\ a + \frac{3}{4}\ log.\ a = \frac{15}{4}\ log.\ a.$

6. $\frac{log.\ [(a-b)^n \sqrt{c^m e^3}]}{(f+g)^{\frac{1}{n}} \sqrt[3]{d^4}} = n\ log.\ (a-b) + \frac{m}{2}\ log.\ c +$
$\frac{3}{2}\ log.\ e - \frac{1}{n}\ log.\ (f+g) - \frac{4}{3}\ log.\ d.$

7. $log.\ x = log.\ (a\,b + c\,d\,e) = log.\ \left[ab\left(1 + \frac{c\,d\,e}{a\,b}\right)\right]$
$= log.\ a + log.\ b + log.\ \left(1 + \frac{c\,d\,e}{a\,b}\right).$

8. $log\ \sqrt[n]{(a^3 - x^3)^m} = \frac{m}{n}\ log.\ (a-x) + \frac{m}{n}\ log.\ (a^2 + a\,x$
$+ x^2) = \frac{m}{n}\ log.\ (a-x) + \frac{m}{n}\ log.\ [\,(a+x)^2 - a\,x] =$ (putting $z^2 = a\,x$) $\frac{m}{n}$ $[log.\ (a - x) + log.\ (a + x + z) +$
$log.\ (a + x - z).]$

9. $log.\ \sqrt{(a^2 + x^2)} =$ (taking $2\,a\,x = z^2$) $\frac{1}{2}$ $[log.\ (a + x$
$+ z) + log.\ (a + x - z)\ ].$

10. $log.\ \frac{\sqrt{(a^2 - x^2)}}{(a+x)^2} = \frac{1}{2}\ [log.\ (a - x) - 3\ log.\ (a + x).]$

11. If required to insert m means by quotient, between a and z, we take $n = m + 2$ in the equation $z = a\,q^{n-1}$ (260), whence $q = \sqrt[m+1]{\frac{z}{a}}$, and $log.\ q = \frac{log.\ z - log.\ a}{m+1}$. The logarithms of the several terms aq, aq^2 . . . . are then, $log.\ a + log.\ q$, $log.\ a + 2\ log.\ q$ . . . Thus, to insert 11 means between 1 and 2, here the $log.\ a = 0$, we find $log.\ q = \frac{1}{12} \times log.\ 2 = 0{,}0250858$ $\therefore$ $q = 1{,}059463$; the log. of the consecutive terms are $2 log.q$, $3\ log.\ q$ . . . . . . and the progression is

1 : 1,059463 : 1,122461 : 1,189207 : . . . . 1,887747 : 2.

12. Be x the unknown quantity in the equation $b^{n-\frac{c}{x}}$

$= c^{mx} f^{x-q}$, we get $\left(n - \frac{a}{x}\right)$ $log.\ b = m\ x\ log.\ c + (x - q)$ $log.\ f$, and the equation of the 2nd degree remains to be solved $(m\ log.\ c + log.\ f)\ x^2 - (n\ log.\ b + q\ log.\ f)\ x + a.log.\ b = 0$

13. $c^{mx} = a\ b^{nx-1}$ gives $x = \dfrac{log.\ a - log.\ b}{m\ log.\ c - n\ log.\ b}$.

14. *The population of a town increases every year* $\frac{1}{30}$*th part; how many inhabitants will it contain after one century, the number being now* 100,000? *Making* $n = 100{,}000$; *after one year the population will be* $n + \frac{n}{30} = \frac{31}{30}\ n = n'$. *After the following year* n' *will also become* $\frac{31}{30}\ n' = \left(\frac{31}{30}\right)^2 n \ldots$ *&c.; then at the end of* 100 *years the number of inhabitants will be*

$log.\ 31 =$	1,49136169
$-\ log.\ 30 =$	1,47712125
	0,01424044
multiplying by 100	1,424044
$log.\ n$	5,000000
$log.\ x$	6,424044

$x = n\left(\frac{31}{30}\right)^{100}$ 2654874

15. If the annual increase be a r^e part of the present inhabitants, we find that the primitive number n, becomes after m years, $x = n\left(\frac{1+r}{r}\right)^m$. We may take here the unknown quantity x, n, r, or m, the other quantities being known,

$$log.\ x = log.\ n + m\ [log.\ (1 + r) - log.\ r],$$
$$log.\ n = log.\ x - m\ [log.\ (1 + r) - log.\ r.]$$

$m \quad \dfrac{log.\ x - log.\ n}{log.\ (1 + r) - log.\ r},\ log.\left(1 + \frac{1}{r}\right) \quad \dfrac{log.\ x - log.\ n}{m}$

Questions relating to the Interest of Money.

CCCXXVI. The principles of progression by quotients and of logarithms will be found to occur in calculations relating to interest. To understand what we have to offer on this subject it must be recollected, that the income derived from a sum of money employed in trade, or in executing some productive work, will be in proportion to the frequency with which it is exchanged in either case. Hence it follows, that he, who borrows a sum of money for any purpose, ought upon returning this money at the expiration of a given time, to allow the lender a premium equivalent to the profits, which he might have received, if he had employed it himself. Such is the view in which the subject of interest presents itself. In order to determine the interest of any sum, we compare this sum with 100 rupees taken as unity, having fixed the premium, which ought to be allowed for this last at the end of a particular term, one year for example. We shall not here consider those things, which, in the different kinds of speculation, occasion the rise and fall of interest; this belongs to the elements of political and commercial arithmetic, which should be preceded by some account of the doctrine of chances. Our object in what follows is simply to resolve certain questions, which refer themselves to progression by quotients.

To present this subject in a general point of view, we shall suppose the annual premium, allowed for a sum 1, to be represented by r, r being a fraction; it is evident, that the interest of a sum 100, for the same time, will be 100 r, that of any sum whatever a will be denoted by $a\,r$; if we designate this last by α, we have

$$\alpha = a\,r.$$

By means of this formula, it is easy to find the interest of any sum whatever, when that of 100 or of any other sum, for a known time, is given; questions of this kind belong to what is called *simple interest*.

CCCXXVII. But if the lender, instead of receiving annually the interest of his money, leaves it in the hands of the borrower to accumulate, together with the original sum, during the following year, the value of the whole at the end of this year may be found in the following manner. The original sum being a, if we add to it the interest $a\,r$, it becomes at the end of the first year

$$a + a\,r = a\,(1 + r).$$

Now if we make

$$a(1+r)=a',$$

the interest of the sum a' for one year being $a'r$, that of the sum $a(1+r)$ will be for a second year, $ar(1+r)$; and as, at the end of the first year, the principal a augmented by the interest, becomes $a(1+r)$, the principal a' amounts at the end of the second year, to

$$a'(1+r)=a(1+r)^2=a''.$$

If the lender does not now withdraw the sum a'', but leaves it to accumulate during a third year, at the end of this, it will become, according to what precedes,

$$a''(1+r)=a(1+r)^3=a'''.$$

It will be readily perceived, that a''' will become at the end of the fourth year

$$a'''(1+r)=a(1+r)^4,$$

and so on, and that, consequently, the sum first lent, and the several sums due at the end of the first, second, third, fourth, &c. years, form the following progression by quotients;

$$\div a : a(1+r) : a(1+r)^2 : a(1+r)^3 : a(1+r)^4 : \text{\&c.}$$

of which the quotient is $1+r$, and the general term

$$a(1+r)^n=A,$$

the number n representing the number of years, during which the interest is suffered to accumulate.

If the rate of interest be 5 per cent., for example, that is, if for 100 rupees during one year 105 rupees are paid back; we have

$$100\,r=5, \quad \text{or} \quad r=\tfrac{5}{100}=,05 \quad \text{and} \quad 1+r=1,05.$$

If we would know to what the sum a amounts, when left to accumulate during 25 years, we have

$$n=25, \quad \text{and} \quad A=a(1,05)^{25}.$$

instead of the original sum. The 25th power of 1,05 may be easily found by means of logarithms, since we have (322)

$$l\,(1,05)^{25}=25\,l\,(1,05)=0,5297322,$$

which gives

$$(1,05)^{25}=3,386 \quad \text{nearly}, \quad A=3,386\,a;$$

and hence it may be readily seen, that 1000 rupees will in this way amount at compound interest to 3386 rupees, at the end of 25 years.

If the sum lent were for 100 years, we should have

$$A = a\,(1{,}05)^{100} = 131\ a$$

nearly; thus 1000 rupees would produce, at the end of this period, a sum of 131000 rupees nearly. These examples will be sufficient to show with what rapidity sums accumulate by means of compound interest.

CCCXXIX. The equation

$$A = a\,(1 + r)^n,$$

gives rise to four questions; the first which is to find A, when a, r, and n, are known, presents itself, whenever we seek the amount of the principal at the end of a number n of years. We have already given an example of this.

The second, which is to find r, when a, A, and n, are known, occurs whenever it is required to determine the rate of interest by means of the original sum, the whole amount that has become due, and the time during which it has been accumulating; we have in this case

$$1 + r = \sqrt[n]{\frac{A}{a}}.$$

The third, which is to find a, when A, r, and n are known, the formula for which is

$$a = \frac{A}{(1 + r)^n},$$

has for its object to determine the principal, which it is necessary to employ in order to be entitled after a number n of years, to a sum A.

The fourth, which is to find n, when A, a, and r are known, can be resolved only by means of logarithms (269, 322). Taking the logarithm of each member of the proposed equation, we have

$$l\,A = l\,a + n\,l\,(1 + r,)$$

whence

$$n = \frac{l\,A - l\,a}{l\,(1 + r)}.$$

By means of this last equation we determine how many years the principal a must remain at interest in order to amount to a sum A.

To illustrate this by an example, we shall suppose that it is required to find the time in which the original sum will be doubled, the rate at interest being 5 per cent.; we have

$$A = 2\,a,\ l\,A = l\,a + l\,2,$$

and consequently,

$$n = \frac{l\,2}{l\,(1,05)} = \frac{0,3010300}{0,0211893} = 14,2067$$

nearly.

CCCXXX. The following question is one of the most complicated, that we meet with relating to this subject. We suppose, that the lender during a number n of years, adds each year a new sum, to the amount of this year; it is required to find what will be the value of these several sums, together with the compound interest that may thence arise at the expiration of the term proposed. Let $a, b, c, d, \ldots. z$, be the sums added the first, second, third, fourth, &c. years; the sum a remaining in the hands of the borrower during a number n of years, amounts to

$$a\,(1+r)^n;$$

the sum b, which remains $n-1$ years only, becomes

$$b\,(1+r)^{n-1},$$

the sum c, which remains $n-2$ years only, becomes

$$c\,(1+r)^{n-2},$$

and so on; the last sum, z, which is employed only one year becomes simply

$$z\,(1+r);$$

we have, therefore,

$$A = a\,(1+r)^n + b\,(1+r)^{n-1} + c\,(1+r)^{n-2} \ldots. + z\,(1+r).$$

By calculating the several terms of the second number separately, we obtain the value of A.

The operation is very much simplified when

$$a = b = c = d \;\ldots\ldots\; = z,$$

for in this case we have

$$A = a\,(1+r)^n + a\,(1+r)^{n-1} + a\,(1+r)^{n-2} \ldots. + a\,(1+r);$$

the second member of this equation forms a progression by quotients, of which the first term is $a\,(1+r)$, the last term $a\,(1+r)^n$, the quotient $1+r$, and the sum, consequently,

$$\frac{a\,(1+r)^{n+1} - a\,(1+r)}{r} \;(261);$$

we have, therefore, in this case,

$$A = \frac{a\,(1+r)\,[\,(1+r)^n - 1]}{r}.$$

This equation gives rise also to four questions corresponding to those mentioned in connexion with the equation

$$A = a\,(1+r)^n.$$

CCCXXXI. By reversing the case we may represent those annual sums, or sums due at stated intervals, called *annuities*; here the borrower discharges a debt with the interest due upon it, by different payments made at regular periods. These payments, made by the borrower before the debt in question is discharged, may be considered, as sums advanced to the lender toward the discharge of the debt, the value of which sums will depend upon the interval of time between the payment and the expiration of the annuity. Thus, if we represent each sum by a, the first payment, which will take place $n - 1$ years before the expiration of the term of the annuity, referred to this time, is worth $a\ (1 + r)^{n-1}$; the second, referred to the same epoch, is worth only $a\ (+ r)^{n-2}$: the third, $a\ (1 + r)^{n-3}$; and so on to the last, which amounts only to the value of a. But on the other hand, the sum lent being represented by A, will be worth in the hands of the borrower, after n years, $A\ (1 + r)^n$, which must be equal to the amount of the several payments advanced by him to the lender; we have, therefore, $A\ (1 + r)^n = a\ (1 + r)^{n-1} + a\ (1 + r)^{n-2} + a\ (1 + r)^{n-3} \ldots + a$, or taking the sum of the progression, (263) which constitutes the second member

$$A\ (1 + r)^n \quad \frac{a\ [(1 + r)^n - 1]}{r}, \therefore A = \frac{a\ [(1 + r)^n - 1]}{r\ (1 + r)^n}$$

an equation, in which we may take for the unknown quantity, successively, the quantity A, which we shall call the *value* of the annuity, because it is the sum, which it represents; the quantity a, which is the *quota* of the annuity; the quantity r, which is the rate of interest, and lastly, the quantity n, which denotes the term of the annuity. In order to find this last, we must have recourse to logarithms. We first disengage $(1 + r)^n$, which gives

$$(1 + r)^n = \frac{a}{a - A\ r},$$

then taking the logarithms, we have

$$n\ l\ (1 + r) = l\ a - l\ (a - A\ r),$$

whence

$$n = \frac{l\ a - l\ (a - A\ r)}{l\ (1 + r)}.$$

CCCXXXII. To give an instance of the application of the above formulæ, we shall take the following question:

To find what sum must be paid annually, to cancel in 12 *years a debt of* 100 *rupees with the interest during that time, the rate of interest being* 5 *per cent.*

In this example, the quantities given are

$$A = 100, \ n = 12, \ r = ,05$$

and the annuity a is required to be found; resolving the equation

$$A\,(1 + r)^n = \frac{a\,[\,(1 + r)^n - 1]}{r}$$

with reference to the letter a, we have

$$a = \frac{A\,r\,(1 + r)^n}{(1 + r)^n - 1}.$$

The value of the letters, A, r, and n, are to be substituted in this expression; and it will be found most convenient in the first place, to calculate by the help of logarithms, the quantity $(1 + r)^n$, which becomes $(1,05)^{12}$; and

$$(1,05)^{12} = 1,79586.$$

By means of this value we obtain

$$a = \frac{100 \times ,05 \times 1,79586}{1,79586 - 1} = \frac{5 \times 1,79586}{0,79586};$$

and, determining the values of this last expression either directly or by means of logarithms, we find

$$a = 11,2826;$$

an annuity of 11,28 rupees, therefore, is necessary to cancel in 12 years a debt of 100 rupees, the rate of interest being 5 per cent.

CCCXXXIII. In order to compare the values of different sums, as they concern the person who pays or receives them, they must be reduced to the same epoch, that is, we must find what they would amount to when referred to the same date. A banker, for instance, owes a sum a payable in n years; as an equivalent he gives a note, the nominal value of which is represented by b, and which is payable in p years, the first sum at the time the note is given, is worth only $\frac{a}{(1 + r)^n}$, because it must be considered as the original value of a principal, which amounts to a at the expiration of n years; the sum b, for the same reason, is worth at the time the note is given $\frac{b}{(1 + r)^p}$; the difference

$$(1 + r)^n \qquad (1 + r)^p$$

represents, therefore, according as it is positive or negative, what the banker ought to give or receive by way of balance; if this balance is not to be paid until after a number of years denoted by q, c representing its value at the time the exchange is made, it will amount at the expiration of this term to

$$c\,(1+r)^q;$$

so that it will be equivalent to

$$\left(\frac{a}{(1+r)^n}-\frac{b}{(1+r)^p}\right)(1+r)^q = a\,(1+r)^{q-n} - b\,(1+r)^{q-p}.$$

The several sums, a, b, ... x, in art. 330, were reduced to the time of the payment of the sum A, and in art. 331, each of the payments, as well as the sum A, was referred to the time, when the annuity was to cease.

CCCXXXIV. The following process for finding the values of annuities is sometimes preferred.

The debt A after the 1st year becomes $A\,(1+r)$, or for shortness sake, Aq, on paying a sum a, the debt is reduced, and is only $Aq - a = A'$. Paying the same sum a after the 2nd year, the debt becomes reduced to $A'\,q - a$, or $A\,q^2 - aq - a$; proceeding in the same manner, we ascertain what is due after any number of years; thus we find that after n years, and having made his nth payment, the debt is reduced to

$$x \text{ (the remaining debt)} = A\,q^n - a\,q^{n-1} \quad a\,q^{n-2} - a\,q^{n-3}$$
$$- a = A\,q^n - a\,(q^{n-1} - q^{n-2} \ldots + 1) \qquad A\,q^n - a\left(\frac{q^n - 1}{q - 1}\right)$$

substituting for the value of $q = 1 + r$, we get

$$x = A\,(1+r)^n - a\left(\frac{(1+r)^n - 1}{r}\right);$$

should the debt be liquidated, we have $x = 0$, or

$$A\,(1+r)^n = \frac{a\,[\,(1+r)^n - 1]}{r}, \text{ as before.}$$

By this theory are ascertained the values of such rents of which the capital and interest terminate with the death of the lender. The lender is supposed to have n years more to live, at the time he lends his capital A, the question being what sum a must be paid to him annually that, at the expiration of these n years, he may have no further claim: this rent or *annuity* is given by the value a (332). If, for example, the perpetual interest of rupees 100 is 5, and the annuitant expected to live 12 years longer, the amount of the annuity is rupees 11,2826.

It is true that we do not know before hand, how long the buyer of an annuity will live, but it can be supposed from the tables of annuities, and though this supposition may be erroneous for one individual, it becomes exact for a great number taken together, because some individuals gain precisely what others lose. We know then the probable life of an individual whose age is known. Taking the following two lines, of which the first denotes the ages. and the second, the number n years which individuals corresponding to the age indicated in the 1st line, have to live

Ages	1,	5,	10,	15,	20,	25,	30,	35,	40,	45,	50,	55,	60,	65,	70,
n years	20,	35,	40,	37,	35,	31,	28,	24,	19,	16,	13,	10,	7,	5,	3.

A man of 45 years of age being supposed to have yet 16 years to live, then (332) $n = 16$, $A = 100$ and r the interest in perpetuity $= 5$ p$\frac{0}{0}$, we find for a the value of the annuity $= 9{,}228$: that is, the capital should be placed at the annual interest of $9\frac{1}{4}$ per cent. nearly.

Questions relating to Interest and Annuities.

1. *A person puts Rs.* 3300 *out at the interest of* 6 *per cent.; at the expiration of each year the interest is added to the principal; and to which at the same time he adds an equal sum. What will it amount to in* 20 *years?*

Ans. *Rs.* 128676.

2. *At what rate per cent. compound interest must a sum of Rs.* 409 *be laid out annually to amount to Rs.* 43258, *in* 35 *years?*

Ans. $5\frac{1}{2}$ *per cent.*

3. *What capital when put to compound interest at* $4\frac{1}{2}$ *per cent. and at the expiration of each year being increased by the same sum on the same conditions, will in* 12 *years amount to Rs.* 88136,24?

Ans. 5454.

4. *How many yearly payments of Rs.* 112 *are required, each placed at compound interest of* $10\frac{1}{2}$ *per cent., to amount to Rs.* 8415,32?

Ans. 21.

5. *A capital of Rs.* 3257.5 *is put out at the interest of* $9\frac{1}{2}$ *per cent. and at the expiration of each year the interest is added to the capital, together with the sum of Rs.* 1250. *What will the several sums amount to in* 18 *years?*

Ans. *Rs.* 67677,7.

6. *A man saved every year from his salary Rs.* 1200, *which he placed at* $4\frac{3}{4}$ *per cent. interest, and having allowed it to accumulate he added each year the same sum during a certain number of years, his capital with compound interest amounted to Rs.* 57966. *After how many years did his capital amount to that sum?*

Ans. 25 *years.*

7. *A person borrows Rs.* 60, *for which he promises to pay* $15\frac{1}{2}$ *per cent., but instead of refunding this amount, goes on borrowing the same sum yearly; at the expiration of the 5th year, the lender refused any further loans; how much did all he had lent amount to, calculating with compound interest?*

Ans. Rs. 471,894.

8. A *and* B *took a life annuity of Rs.* 3500 *each, interest being calculated at* $3\frac{5}{8}$ *per cent. But* B *being* 10 *years younger than* A, *had to pay Rs.* 12831,4 *more than* A. *What was the respective ages of* A *and* B?

Ans. 30 *and* 40 *years.*

9. A *and* B *buy each an annuity* a, *when interest is reckoned at* r *per cent.* A*'s life being calculated of* m *years greater duration than* B*'s, whose life is worth* n *years, what is the expression of the difference of the values of their annuities?*

Ans. $\dfrac{a\,[\,(1+r)^m - 1\,]}{r\,(1+r)^{m+n}}$.

10. A*'s age being required from the circumstance of his having bought an annuity of Rs.* 3500, *for Rs.* 59457,9, *interest being calculated at* $3\frac{5}{8}$ *per cent.*

Ans. 30 *years.*

11. *How long must a capital stand at* $3\frac{3}{4}$ *per cent. that it may double itself; and how long that it may triple itself?*

Ans. Doubles in $18{,}829$ *years; triples in* $29{,}842$ *years.*

12. *How long must a capital stand at* r *per cent. that it may be* m *times as great?*

Ans. $\dfrac{\log.\, m}{\log.\,(1+r)}$.

13. *If a principal* x, *be put out at compound interest for* x *years, at* x *per cent., required the time in which it will gain* 4 x.

Ans. 13,08659 *years.*

14. *An old man* 70 *years of age, finding the annual interest of* $4\frac{1}{2}$ *per cent. of his capital of Rs.* 16493,8 *insufficient*

to live in comfort, bought an annuity for that capital. What is the amount of the annuity?

Rs. 6000.

15. A *lends Rs.* 3030 *to* B *at the rate* $7\frac{2}{3}$ *per cent. Striking the balance of their account every year, and adding the same sum to it at the expiration of each year,* B *finds that after a certain number of years the sum to his debit was Rs.* 53346. 12 *anas. After how many years did he owe that sum?*

Ans. 11 *years.*

16. *A person obtains an annuity of Rs.* 2000,—*for Rs.* 30781,11 *interest being calculated at* $5\frac{1}{2}$ *per cent. How old was he?*

Ans. 5 *years.*

17. *A person* 35 *years of age, buys an annuity of Rs.* 2400, *interest being calculated at* $4\frac{1}{2}$ *per cent. What is the present value of it?*

Ans. Rs. 34789. 2 *ans.* 3 *pie.*

18. *An individual bought an annuity of Rs.* 6000, *for Rs.* 47474, *interest being calculated at* $4\frac{1}{2}$ *per cent. How old was he?*

Ans. 55 *years.*

19. *Four planters* A, B, C, *and* D, *bid at an auction for an indigo concern.* A *offers to pay Rs.* 5555,5 *in cash and equal instalments at the expiration of each year for the next five years;* B *offers to pay Rs.* 20000 *in cash, and an equal sum after seven years;* C, *who is* 55 *years old, offers his annuity of Rs.* 5300 *in payment; lastly,* D, *offers to pay Rs.* 12000 *in cash, and after the expiration of each year for the next five years, Rs.* 9000. *Reckoning the interest at* $6\frac{2}{3}$ *per cent. what is the present value of each offer?*

Ans. A's *offer* = *Rs.* 28538,9; B's = *Rs.* 32730,02; C's = *Rs.* 37805,34; D's = *Rs.* 49216,4.

20. *A debt due at the present time amounting to Rs.* 6000, *is to be discharged in* 11 *years by equal payments. What is the amount of these payments if the interest be calculated at* $6\frac{1}{2}$ *per cent.?*

Ans. Rs. 783,692.

21. *If an annuity of Rs.* 1500 *having* 11 *years to run, be worth Rs.* 10783,3, *at what rate is the interest calculated?*

Ans. $6\frac{1}{2}$ *per cent.*

22. *How old is the person, who has to pay Rs.* 12116,23 *for an annuity of Rs.* 800 *on his life, the interest being calculated at* $5\frac{1}{2}$ *per cent.?*

Ans. 25 *years.*

23. *A merchant borrowed from Government Rs.* 20000, *and for this sum mortgaged a house, which yielded a clear annual rent of Rs.* 1500. *How long can government retain the title-deeds of the house, if the interest be calculated at* 5 *per cent.?*

Ans. 22 *years nearly.*

24. *A person has put his whole fortune of one lac of rupees, out at* $4\frac{1}{2}$ *per cent. interest; but this not being sufficient to cover his expenses, which amount to Rs.* 6000 *yearly, he is obliged at the end of each year, to take as much from his capital as will, with the interest he receives, make up Rs.* 6000. *In how many years will he become a beggar?*

Ans. 31,4946 *years.*

25. *A usurer lent a person Rs.* 600, *and drew up for the amount a bond for Rs.* 900, *payable in* 4 *years, without interest. What did he take per cent. compound interest being taken into consideration.?*

Ans. above $10\frac{2}{3}$ *per cent.*

26. *A usurer lent Rs.* 900, *and drew up, for the amount, a bond of Rs.* 1900, *payable in* 5 *years, bearing compound interest at* 4 *per cent. How much did he take per cent. calculating compound interest?*

Ans. Somewhat above $20\frac{3}{4}$ *per cent.*

27. *What is the present value of an annuity of Rs.* 400 *for* 30 *years, payable quarterly, when the interest is convertible into principal monthly, the rate of interest being* 6 *per cent.?*

Rs. 5532. 0 *anas.* $11\frac{5}{8}$ *pie.*

28. *What is the present value of an annuity of £* 40 *for* 30 *years, payable quarterly, when the interest is convertible into principal half-yearly, the rate of interest being* 4 *per cent.?*

£698. 13*s.* $10\frac{1}{2}$*d.*

29. *To what sum will an annuity of Rs.* 100, *payable annually, amount in* 20 *years, when laid up and improved at compound interest, the interest at the rate of* $4\frac{1}{2}$ *per cent. per annum, being always converted into principal twice a year?*

Rs. 3154.

30. *What is the present value of a perpetual annuity of* £100, *receivable in equal half-yearly payments, when the interest, at the rate of* 5 *per cent. per annum, is convertible into principal monthly?*

£1979. 6*s.* $8\frac{3}{4}$*d.*

31. *If a capital* A *is put out at* r *per cent. interest, and a sum* a *is annually taken away, greater than the interest of the capital* A, *in how much time will the capital be spent?*

$$\textit{Ans.}\quad \textit{Number of years } n = \frac{l\,a - l\,(a - A\,r)}{l\,(1 + r)}.$$

32. *If a capital* A *is lent out at the rate* r *per cent. in what time will it become* A′, *if the capital, with the interest and compound interest added to it, be augmented or diminished yearly by the sum* a?

$$\textit{Ans.}\quad n = \frac{l\,(A'\,r \pm a) - l\,(A\,r \pm a)}{l\,(1 + r)}.$$

The upper sign of $\pm$ denotes the addition of a, the lower the deduction of a.

33. *What is the present value of an annuity of Rs.* 100, *to continue* 25 *years, the rate of interest being* 4 *per cent., payable yearly? half-yearly? quarterly? monthly?*

Ans.	*Yearly*	*Rs.*	1562,21.
	Half-yearly	,,	1571.18.
	Quarterly	,,	1575,73.
	Monthly	,,	1578,81.

34. *What is the present value of an annuity of Rs.* 40, *to continue* 20 *years, but not to commence till the end of* 5 *years, at* 4 *per cent.?*

Ans. Rs. 446. 13 *anas, nearly.*

35. A *orders* B *to purchase an Indigo factory and to debit his account with it, charging him at the usual rate of interest of* $7\frac{2}{3}$ *per cent. At the end of each year,* A *remits him Rs.* 1235,88 *on account, for a certain number of years till his debt was cancelled. Had he made his annual payments of Rs.* 1585, *he should have cleared his debt five years earlier. What was the total outlay of the concern?*

Ans. Rs. 10797,26.

36. A *borrows from* B, *a sum of money at the rate of* r *per cent. At the expiration of each year he pays on account* b *rupees, for* x *years till the debt was cleared. Had he paid* a *rupees he should have cleared his debt* m *years earlier. What sum did he borrow?*

$$\textit{Making } q = \frac{a}{b};\ \textit{Ans. } x = \frac{l\,[q\,(1 + r)^m - 1] - l\,(q - 1)}{l\,(1 + r)} - m$$

$$\therefore A = \frac{b\,[(1 + r)^x - 1]}{r\,(1 + r)^x}.$$

Miscellaneous Problems.

1. *In a rectangular piece of ground* a *feet long and* b *broad, there are two diagonal walks crossing each other and both of equal width throughout, and the area of the ground taken up by both the walks, is equal to half that of the whole piece of ground. Express the width of the walk in terms of the length and breadth of the piece of ground.*

Ans. $\dfrac{a\,b}{\sqrt{a^2 + b^2}}\,(1 \pm \sqrt{\tfrac{1}{2}})$.

2. *A triangle whose sides are as the numbers* 1, 2, 3, *and whose greatest angle is twice that of the least. What are the sides and angles?*

Ans. Sides 4, 5 *and* 6; *angles* $\begin{cases} 41^\circ\ 24'\ 23''. \\ 55^\circ\ 46'\ 51''. \\ 82^\circ\ 48'\ 46''. \end{cases}$

3. *An engineer officer had observed the angle of the height of a tower on a plane level with the foot of the tower; on advancing in a straight line* 60 *feet towards it, he had there found the angle to be the complement of the former, and after again advancing* 20 *feet in the same direction the angle of elevation was double that of the first. The actual observation being lost, it is required from the above data, to find the height of the tower, the observer's eye being* 5,84 *feet above the plane.*

Ans. 80 *feet.*

4. *An artillery officer erecting a battery at a certain distance from a fort within the range of his guns, measured with a sextant the angle subtended by two steeples* A *and* B *in the fort, and then that of* B *with a tower* C; *which angles, after reducing them to the horizon, he found* 33° 57′ 30″,2 *and* 40° 11′ 37″,4 *and by means of a map of the fort, he ascertained the distance between* A *and* B *to be* 129311,3 *feet, and that between* B *and* C 130828,5 *feet. Required the distance of the battery from the middle point* B.

Ans. 194944,5 *feet.*

5. *In a triangle where* s *the sum of two sides,* S *the sum of two angles, and* b *the base, are given; express one of the sides in functions of the known quantities.*

Ans. $x = \sqrt{\dfrac{s^2}{2} \pm \dfrac{1}{2}\sqrt{s^4 + \left(\dfrac{s^2 - b^2}{\cos.\ B}\right)^2}}$.

6. *The two sides of a right-angled triangle are* $3x^x$ *and* x^{3x} *feet, and the length of a line bisecting the right angle*

and meeting the hypothenuse is x^{2x} *feet. Determine from these data the area of the triangle.*

Ans. 188,03074 *feet.* *or* 0,9693 *foot.*

7. *Standing on a level with the foot of a tower, and taking with a sextant the angle* S *subtended by the distance between an upper and lower loophole of a tower whose distance is* b *feet, and the distance of the lower window from the foot of the tower* a *feet. Express the distance, from the tower, at which the angle was measured; my eye being* d *feet above the level at the foot of the tower.*

$$Ans.\quad Distance = \frac{2\,(a-d)\,(a-d+b)\,tan.\,S}{b \pm \sqrt{b^2 - 4\,(a-d)\,(a-d+b)\,tan.^2 S}}$$

8. *It is required to find the 3 sides of a right-angled triangle from the following data. The number of square feet in the area is equal to the number of feet of the 3 sides; and the square described on the hypothenuse is less than the square described upon a line equal in length to the two sides, by half the product of the number represented by the base and the area. Required the 3 sides of the triangle?*

Ans. 6, 8, 10.

9. *What is the area of that right-angled triangle whose base, perpendicular, and hypothenuse are denoted by* x^x, x^{2x}, *and* x^{3x}?

Ans. 1,029085 *nearly.*

10. *If the perimeter of a triangle be denoted by* p, *and the three perpendiculars let fall from the angles upon the opposite sides by* a, b, *and* c; *what are the expressions for the sides? Let* $2\,p = a + b + c$.

$$Ans.\quad \frac{p\,a\,b}{a\,b + a\,c + b\,c}\qquad \frac{p\,a\,c}{a\,b + a\,c + b\,c}\qquad \frac{p\,b\,c}{a\,b + a\,c + b\,c}.$$

11. *Knowing* a, b *and* c *to be the sides of a triangle, what is the expression of its area* T?

$$Ans.\quad T = \sqrt{p\,(p-a)\,(p-b)\,(p-c)}.$$

12. *If* a, b, *and* c *denote the sides of a triangle, what is* r *the radius of its inscribed circle, and R that of its circumscribed circle?*

$$r = \sqrt{\frac{1}{p}\,(p-a)(p-b)(p-c)}\,;\quad R = \frac{a\,b\,c}{4\sqrt{p(p-a)(p-b)(p-c)}}.$$

13. *What is the height* h *of a triangle, of which* b *is the base,* a *and* c *the two other sides?*

$$\textit{Ans.}\quad h = \frac{2}{b}\sqrt{p\,(p-a)\,(p-b)\,(p-c)}.$$

14. *Knowing two sides* b *and* c *of a triangle and its included angle* A, *express the area* T *of the triangle.*

$$\textit{Ans.}\quad T = \frac{b\,c\,\sin.\,A}{2}.$$

15. *Determine the area* Q *of a quadrilateral figure whose diameters are* D *and* d, *and the angle of their intersection denoted by* E.

$$\textit{Ans.}\quad Q = \frac{D\,d}{2}\sin.\,E.$$

16. *Express the area of a quadrilateral figure inscribable in a circle, the 4 sides being* a, b, c *and* d.

$$Q = \sqrt{(p-a)\,(p-b)\,(p-c)\,(p-d)}.$$

17. *Express in function of the 4 sides of a quadrilateral figure the angle* ϕ *between any two sides, as the angle between* a *and* b, *its perimeter* p *being* $= \frac{a+b+c+d}{2}$.

$$\textit{Ans.}\quad \tan.\,\frac{\phi}{2} = \sqrt{\frac{(p-a)\,(p-b)}{(p-c)\,(p-d)}}.$$

18. *There is a tank which can be filled by two different aqueducts* A *and* B. *After letting the first run for* $\frac{2}{3}$ *of the time the second would have been in filling it alone, the stream of water by* A *being diverted from it, the second alone was made to flow into the tank until it was full. If both* A *and* B *had been allowed to flow into the tank at the same time, it would have been filled two days sooner, and twice the quantity of water would have flown from the latter aqueduct than in the first case. Required—the time each aqueduct would alone have filled the tank.*

Ans. *A in 6 days, B in 3 days.*

19. *Determine the area of a trapezium, of which the four sides* a, b, c *and* d *are known. Let* b *and* d *denote the two parallel sides, their difference* b — d = f, *and half the perimeter of the triangle* $\frac{a+c+f}{2} = p$.

$$\textit{Ans.}\quad \textit{Area} = \frac{b+d}{f}\sqrt{p\,(p-a)\,(p-c)\,(p-f)}.$$

20. *Given the side* a, *of a regular polygon, and* n *the number of its sides, express its area* S. Q *denoting the quadrant.*

$$Ans. \quad S = \frac{1}{4} na^2 cot. \frac{2Q}{n}.$$

21. *Find the area* S *of a circle of which* r *the radius is known.*

$$Ans. \quad S = \pi r^2.$$

22. *Determine the area* s *of a sector of which the arc has* n *degrees, and the radius of the circle is* r.

$$Ans. \quad s = \frac{n \pi r^2}{360}$$

23. *Required the area* s *of a segment of which the arc has* n *degrees, and* r *is the radius.*

$$Ans. \quad Area \ of \ the \ segment \ s = \frac{r^2}{2} \left(n \frac{\pi}{180} - sin. \, n \right).$$

24. *Express the area of an ellipse when its two semi-axes are* a *and* b, *or when* a, *and* e *its eccentricity, are known.*

$$Ans. \quad Area \ of \ ellipse \ s = \pi \ a \ b = \pi \ a \sqrt{a^2 - e^2}.$$

25. *Determine the area of a sphere* Σ, *of which the radius* ρ *is known.*

$$Ans. \quad \Sigma = 4 \pi \rho^2.$$

26. *Express the area of a zone between two known parallels of latitude* L *and* l *of the same name,* L > l, *the radius of the sphere being* ρ.

$$Ans. \quad \Sigma = 4 \pi \rho^2 \, sin. \left(\frac{L - l}{2}\right) cos. \left(\frac{L + l}{2}\right).$$

Should L and l be of different names, that is, the one being north and the other south latitude, then

$$\Sigma = 4 \pi \rho^2 \, sin. \left(\frac{L + l}{2}\right) cos. \left(\frac{L - l}{2}\right).$$

27. *Express the area of a zone between two given parallels of latitude* L *and* l *and between two given meridians* M *and* m.

$$Ans. \quad \Sigma = \ ^2 \frac{M - m}{90^\circ} \, sin. \left(\frac{L \mp l}{2}\right) cos. \left(\frac{L \pm l}{2}\right)$$

28. *The angles of a triangle are as* : : m : n : q, *and* p *the perpendicular from the greatest angle* m ; *it is required to express* s *the area of the triangle.*

$$Ans. \quad s = \frac{p^2}{2} (cot. \, n + cot. \, q).$$

29. *If through any point* P *within a triangle, three straight lines be drawn from the angles* A, B, C, *meeting the opposite sides in* D, E, F, *then will*

$$\frac{P\,D}{A\,D} + \frac{P\,E}{A\,E} + \frac{P\,F}{A\,F} = 1.$$

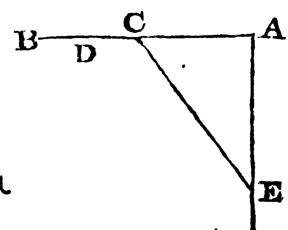
30. *At* 425 *feet distance from an obelisk, whose height is* 357 *feet, a colossal statue on the top of it subtends the same angle to the observer whose height is* $5\frac{3}{4}$ *feet, as a man of the same height standing at the foot of the obelisk. The obelisk and the observer being on the same horizontal plane and the height of the observer's eye* 5,4 *feet: it is required to express the height of the statue.*

Ans. 9,6924 *feet.*

31. *Let the base* AC *and the difference of the hypothenuse* B C *and the perpendicular* A B = d, *of a right angled triangle* ABC, *be given: it is required to determine the hypothenuse.*

Ans. $\textit{Hypothenuse} = \dfrac{b^2 + d^2}{2\,d}.$

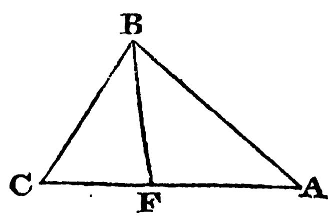
32. *A balcony of pieces of planks of wood,* 12 *feet long and* 4 *feet* 2 *inches broad* (A B) *and* 2 *inches thick, of which the specific gravity is* 0,7, *projecting from a house, is supported by two props as* (C E) $4\frac{1}{2}$ ~~*inches*~~ feet *long of wrought iron, the thickness of each being one inch square. At* D, 9 *inches from the extremity* B *of the balcony, a man weighing* 120 *lbs. is standing.* AC = 3 feet *What will be the total horizontal pressure, against the house at* E *by the two props?*



Ans. 375,4 *lbs.*

33. *Demonstrate that in any triangle* A B C, $AB^2 \times CF + BC^2 \times AF - BF^2 \times AC = AF \times CF \times AC$.

F *is any point in the line* A C.



34. *If the weight of a body sliding freely by means of a thread and a moveable pulley, (the weight of the pulley included,) be expressed by* W, *makes with the two tacks* A *and* B

in the same horizontal line, an angle C, *what is the tension of the thread or what power* P *will keep it in equilibrio?*

$$Ans.\quad P = \frac{W}{2\cos.\frac{C}{2}}$$

35. *Supposing the tacks at* A *and* B, *a feet asunder, not in the same horizontal line as in the preceding example, but making an angle of* d° *with the horizon, and the thread to be* b *feet long, what will then be the expression of the angle at* C *where the weight is suspended by a moveable pulley?*

$$Ans.\quad \sin.\frac{C}{2} = \frac{a}{b}\cos.\ d^{\circ}.$$

36. *Determine the area* Σ *of a trapezium of which two parallel sides* a *and* b, *and their distance* h *between them, is known.*

$$Ans.\quad \Sigma = \frac{a+b}{2}h.$$

37. *Express the area* S *of a triangle* A B C, *knowing one side and its two adjacent angles.*

$$Ans.\quad S = \frac{a^2}{2},\ \frac{\sin.B\sin.C}{\sin.(B+C)}.$$

38. *Divide the triangle* A B C, *into three parts in the ratio* : : m : n : p, *by two lines, drawn from the angle of the summit to the base. Let* x *be the distance from* A, *the extremity of the base* A *to the point where the* 1st *line meets the base, and* y *the distance from this last point to that where the* 2nd *line meets the base, then*

$$x = \frac{m\,b}{m+n+p},\qquad y = \frac{n\,b}{m+n+p}.$$

39. *Divide the triangle* A B C *into two parts* : : m : n, *by a line parallel to one of its sides. Let* A C *be the base, and* D *the point where the line to be drawn parallel to it meets the side* A B, *then*

$$B\,D = A\,B\sqrt{\frac{n}{m}}.$$

40. *Divide the triangle* A B C *into two parts* : : m : n, *by a line* E F *perpendicular to the base* A C. *Be* E *the point on the base* b, *where the perpendicular* E F *meeting one of the sides is to be erected. Call* A E = x.

$$Ans.\quad x = \sqrt{\frac{m\,a\,b}{m+n}}.$$

. . *Should* $m = n$, *then* $n = \sqrt{\frac{a\,b}{2}}.$

41. *An* $8\frac{1}{2}$ *lb. shot and another of* 50 *lbs., both of cast-iron, of which the hemispheres only were above ground and their centres at* $65\frac{1}{2}$ *feet distance from each other; standing on a table placed within the line joining their centres; the height of my eye above ground was* 8 *feet* 11 *inches, both balls appeared of the same magnitude. At what part of the line was I standing?*

Ans. 22,478 *feet from the lesser ball.*

42. *A light being placed* 5 *feet* 9 *inches above the floor, and a sphere of* 6 *inches diameter whose centre is* 8 *inches from the light, and* 5 *feet* 3 *inches above the ground. It is required to find the area of the shadow cast by the sphere.*

Ans. 42,79513 *square feet.*

43. *If two sides of a parallelogram be* a *and* b, *making an angle* d *between them, what is the expression of the resultant* R?

$$R = \sqrt{a^2 + b^2 + 2\, a\, b \cos. d.}$$

44. *If the side of a pentagon inscribed in a circle be* 1 *determine its radius.*

Ans. $\dfrac{\sqrt{5 + \sqrt{5}}}{\sqrt{10}}$.

45. *Let a vessel whose top and bottom are square* (b^2) *and the depth to be* h *feet, be filled with water. It is required to determine the time* (t) *in which the vessel would be emptied through a circular aperture in the bottom whose radius is* a.

Ans. $t = \dfrac{b^2 \sqrt{h}}{a^2 \pi \sqrt{g}}$.

46. *Supposing that in the vessels of the preceding example,* h = 11 *feet* 9 *inches;* b = *feet* 8 *inches;* a = $\frac{1}{4}$ *inch. What will be the time of emptying?*

Ans. 7*h.* 54*m.* 26,5*s.*

47. *If the same vessel were constantly kept full, in what time would a quantity of water equal to the contents of the vessel, flow through the orifice?*

Ans. 3*h.* 57*m.* 13*s.*

48. *If a cone be immersed in a fluid with its vertex downwards determine what part of the axis* (h) *will be immersed if the specific gravity of the fluid is to that of the cone* :: 8 : 1.

Ans. $\dfrac{h}{2}$.

49. *What quantity of water would flow in a given time from a vessel constantly kept full, through a rectangular aperture, in the side of the vessel reaching the top,* 3 *feet deep and* 6 *wide?*

Ans. 117,88 *cubic feet per second.*

50. *What is the quantity of water that would flow per second, from a reservoir constantly kept full, through a rectangular aperture, in the side of the vessel two feet below the top,* 1 *foot deep and* 6 *feet wide?*

Ans. 53,71 *cubic feet.*

51. *What will be the ascending force of a spherical balloon of* $25\frac{1}{2}$ *feet radius filled with a gas having* $\frac{1}{5}$ *the specific gravity of air, of which the cubic foot weighs* $1\frac{2}{9}$ *ounces made of taffeta or thin silk of which the square foot weighs one ounce?*

Ans. 1499 *lbs.* 3 *ounces.*

52. *A sphere of gold having* $1\frac{1}{2}$ *inch radius is to be beaten into a sheet of* $\frac{1}{100}$ *inch thickness and* 14 *inches broad, how long will it be?*

Ans. 8 *feet* 4,98 *inches.*

53. *A cone of silver* 6 *inches high, whose base has* 4,25 *inches radius is to be drawn into a wire* $\frac{1}{80}$ *of an inch radius, how long will it be?*

miles. feet. inches.
Ans. 3 3426, 8.

54. *Letting a stone fall into a well, I observe, that the sound of the stone when it struck the bottom, reached my ear after* 5 *seconds. What was the depth of the well, taking sound to travel* 1150 *feet per second?*

Ans. 354,12 *feet.*

55. *If a diameter of a cylindrical vessel be* 16 *inches, what is its depth, the pressure on the bottom and the sides being equal when filled with a fluid?*

Ans. 8 *inches.*

56. *Suppose that at the same moment a body at rest begins to fall from the point* D, *another body is projected upward from* B *with the velocity due to the height* B C; *it is required to find the point* M *at which the two bodies would meet. Let* C B = 225 *feet,* D B = 190, *then* C D = 225 — 190 = 35 *and* D M = x.

C
D
M
B

$$Ans.\quad x = \frac{190^2}{4 \times 225} = 40\tfrac{1}{9}.$$

45. *Under what angle must a gun be discharged, so that the ball be thrown to the greatest possible horizontal distance? No allowance being made for the resistance of air.*

Ans. 45°.

46. *In a pair of scales a body weighs* 24 *lbs. in one scale, and only* $10\frac{2}{3}$ *lbs. in the other; required its true weight, and the proportion of the lengths of the two arms of the balance?*

Ans. *True weight* = 16 *lbs.*

Proportion of the length of the arms 2 ; 3.

47. *A tapering beam of timber* 32 *feet long, being balanced over a prop when it is* 15 *feet from the greater end, but removing the prop a foot nearer to the less end, it takes* 220 *lbs. suspended from the less extremity, to hold it in equilibrium. Required the weight of the beam.*

Ans. 3520 *lbs.*

48. *A cube of wood sinks* $\frac{2}{3}$ *in sea water, and* $\frac{4}{10}$ *of an inch deeper in fresh water. What is its magnitude, its specific gravity and its weight?*

Ans. Side of the cube = 20 *inches; specific gravity* $0{,}68\frac{2}{3}$; *weight* 198.69 *lbs. avoirdupois.*

49. *A mass of fine gold balances a mass of copper when suspended in air from the equal arms of a lever; determine what is the specific gravity of the fluid in which they balance when the length of one of the arms is made double that of the other?*

Ans. = 5, 86 *nearly.*

50. *If an upright vessel filled with water and* h *feet high, stands on an horizontal plane: at what distance on that plane will the water spout through an orifice in the side of the vessel at the distance of* m *feet from the top of the vessel?*

Ans. $= 2\sqrt{hm}$.

51. *What is the greatest horizontal distance to which water can spout, and where must the orifice be to spout to the greatest distance?*

Ans. Equal to the height of the vessel. The orifice must be applied at half the height of the vessel.

52. *What will be the lateral pressure of an upright cylindrical vessel whose perimeter is* p, *and whose altitude is* h?

Ans. $\frac{h^2 p}{2}$.

53. *If the vessel be a cube, what will be the lateral pressure on each side?*

Ans. $\frac{h^3}{2}$

54. *If in any vessel there be strata* h, h′, h″, *&c. of fluids, whose densities are* d, d′, d″ = *determine the pressure on the base* b.

Ans. $= b\,(h\,d + h'\,d' + h''\,d'' + \&c.)$

55. *If any side of a vessel be oblique to the horizon, what will be the pressure of the fluid on that side? Taking the distance of the surface from the centre of gravity of the side to be* m *feet, and* s *the specific gravity of the fluid, also let the area of the side be expressed by* a^2?

Ans. $a^2\,m\,s$.

56. *Let a vessel whose top and bottom are squares, of which the side is* $5\frac{2}{3}$ *feet, and the depth* $11\frac{3}{4}$ *feet, having a circular orifice whose radius is* $\frac{1}{4}$ *inch, be filled with water. In what time would all the water pass through the orifice? Taking according to Newton, velocity at the vena contracta : velocity at the orifice : :* $\sqrt{2} : 1$.

Ans. 7*h.* 54*m.* 26*s*, 5.

57. *If a ball be discharged from a mortar, making an angle of* e *degrees with the horizon, with an initial velocity* v, *what is the expression of its amplitude, of the greatest height the ball will ascend? and of the time required to strike the ground? Let* g *represent the force of gravity,* t *the time, no allowance for the resistance of air being made.*

Ans. Amplitude $= \frac{v\ \sin.\,2\,e}{g}$; greatest height $= \frac{v^2\ \sin.^2 e}{2\,g}$

time required $= \frac{2\,v\ \sin.\ e}{g}$.

58. *The same data as in the preceding question with the additional condition, that the cannon ball was discharged from an eminence of* m *feet above ground. Required the amplitude, the greatest height and the time required to describe that amplitude.*

Ans. Amplitude

$$= \sqrt{\frac{2\,v^2\ \cos.^2 e}{g^2}\left(v^2 \sin.^2 e + v \sin. e\sqrt{v^2 \sin.^2 e + 2gm} + gm\right) + m^2};$$

time to describe that distance $= \dfrac{v\ sin.\ e + \sqrt{v^2\ sin.^2\ e + 2\ m\ g}}{g}$

greatest height $= \dfrac{v^2\ sin^2\ e + 2\ m\ g}{2\ g}$.

59. *The same data as in* (57) *with the additional condition that the object the ball is to strike, is situated on a hill* m *feet high above the level of the mortar.*
Amplitude

$$= \sqrt{\frac{2v^2 cos.^2 e}{g^2}} \left(v^2 sin.^2 e + v\ sin.e \sqrt{v^2 sin.^2 e - 2gm - gm}\right) + m^2$$

$t = \dfrac{v\ sin.e \mp \sqrt{v^2 sin.^2 e - 2mg}}{g}$; *greatest height* $= \dfrac{v^2 sin.^2 e - 2gm}{2\ g}$.

60. *Being commanded to direct a battery to a certain building in the enemy's fort, knowing its horizontal distance to be* 70623,4 *feet, and observing the angle of elevation to be* 19° 30′, *what elevation must I give to my pieces of ordnance to hit it?*

Ans. 65° 45′ 40″.

61. *What is the centrifugal force of an object situated on the surface of the earth's equator?*

Ans. 0,11117.

62. *How many times must the centrifugal force be greater than it actually is, in order to balance the attraction of gravity, so that an object close to the surface of the earth, would have no tendency to fly off nor to fall?*

Ans. 289.

63. *If a hard or soft body* A *wholly inelastic, impinges on another* B *of a different mass but also inelastic, moving in the same or opposite direction, what will be the velocity of each body after contact? expressing the velocity of the first body by* v *and its mass by* m, *the second body* B *by* v′ *and its mass by* m′, *of then the common velocity* V A *and* B *after contact is?*

Ans. $V = \dfrac{m\ v \pm m'\ v'}{m' + m}$.

64. *Supposing a body to strike another body at rest but of a different mass, with what velocity does it move after contact, the data remaining the same as in the former question?*

Ans. $V = \dfrac{m\ v}{m' + m}$.

65. *If the body* B *is at rest and of an infinite magnitude compared with that of* A *in motion, were struck by* A, *what*

would be the velocity of A, *and that of* B *after impact, both bodies being elastic?*

Ans. A returns with the same velocity it had at first; B remains at rest.

66. *If an elastic body* A *strikes against another elastic body* B *of a different mass, moving in an opposite direction what will be the velocity of* A *and of* B *after contact? Representing the velocity of* A *after impingement by* u, *and that of B by* u', *then*

$$u = \frac{m\,v + m'\,(2\,v' - v)}{m' + m}\ ;\ u' = \frac{m'\,v' + m\,(2\,v - v')}{m' + m}.$$

67. *Should the masses of* A *and* B *be equal, viz.* m = m′ *what will be the velocity of each after impact?*

Ans. $u = v'$ *and* $u' = v$.

68. *Should* B *be at rest but not equal to* A, *what is the velocity of* A *after impact, and what that of* B?

Ans. $u = \frac{2\,(m - m')}{m + m'}\ ;\ u' = \frac{2\,m\,v}{m + m'}$.

69. *Should* B *be at rest and* A *infinitely great compared to* A, *what will be the velocity of* A *and that of* B *after impingement?*

Ans. $u = -v$, *and* $u' = o$.

70. *Should* B *be at rest and equal in mass to* A, *what is the velocity after the impingement of* A *and* B?

Ans. $u = o$, *and* $u' = v$.

71. *If a cannon ball weighing* 25 *lbs. penetrating a balistic pendulum weighing* 7475 *lbs. gives it a velocity of* 2 *feet per second, what was the initial velocity of the cannon ball?*

Ans. 1500 *feet.*

72. *The force of gravity is in the latitude of London* = 32,1908 *feet, what must be the length of a pendulum there to vibrate seconds?*

Ans. 39,13929.

73. *At what angle must a plane be inclined so as to counterbalance a weight* a, *resting on an inclined plane connected by a string over a pulley with another weight* b *hanging freely over the perpendicular?*

Ans. sin. *of inclined plane* $= \frac{b}{a}$.

74. *If a man can draw a weight of* 100 *lbs. up the side of a perpendicular wall* 12 *feet high, what weight will he be able to raise on an inclined plane* 36 *feet long laid aslope from the top of the wall, the resistance from friction on the inclined plane being equal to* ¼ *of the weight so raised?*

Ans. 240 *lbs.*

75. *Two bodies weighing* M *and* m *lbs. are suspended over a pully by means of a string; ascertain how far the greater will descend in* t *seconds of time.*

Ans. Space $= \dfrac{g\,m\,t^2}{2\,(M+m)}$.

76. a *inches from one end of a cylindrical pole is suspended a weight of* b *lbs.; the pole is* c *feet long, weighing* d *lbs. How far from that end is the centre of gravity?*

Ans. $= \dfrac{d\,c + 2\,a\,b}{2\,(b+d)}$.

77. *If a sphere 9 inches in diameter, sinks 6 inches in pure water, what is its specific gravity and weight?*

Ans. *Specific gravity* =, 7407; *weight* 10,227 *lbs.*

78. *A beam of wood when supported horizontally, can at a certain point just bear* a *lbs. without breaking; how much will it be able to support at the same point, when inclined at an angle with the horizon of* d^0?

Ans. $\dfrac{a}{\cos.\,d}$.

The difference between the equatorial and polar weights of the same mass being : : 229 : 230, and the rate of increase of weight from the equator to the pole *being as the square of the sine of latitude:*

79. *What must be the length of the pendulum vibrating seconds in Calcutta, latitude* 22° 33′, *that of London,* (*latitude* 51° 30′) *being* 39.13929 *inches?*

Ans. 39.06003 *inches.*

80. *If a pendulum vibrating in an arc of* 13° 37′ 30″ *be* 39.06003 *inches long, what is its velocity at the lowest point, taking the force of gravity* $32\frac{1}{6}$?

Ans. 14,1785 *inches.*

81. *If the distance from the point of suspension to the centre of oscillation of a pendulum be* 3 *inches, how many vibrations will it perform in a minute?* ($g = 32\frac{1}{6}$.)

Ans. 237,3.

82. *What must be the length of a pendulum, in Calcutta, to vibrate only* 25 *times in a minute?*

Ans. 6,7813 *inches.*

83. *In what latitude will a pendulum* 39,12 *inches long vibrate seconds, and what is the force of gravity at that place?*

Ans. *Latitude* 44° 56′ 35″.
$g = 32{,}175$ *feet.*

84. *Required to find the length of a half second pendulum, that is, a pendulum vibrating twice in a second at Calcutta.*

Ans. 9,675 *inches.*

85. *Two weights are maintained in equilibrio on two differently inclined planes* A B *and* C B, *by means of a string passing over a pulley at* B, *where the two planes, each being* 39,6⅓ *inches high, meet. The pressure on* C B *caused by a weight of* 36 *lbs. being* 1, *what is the pressure on* A B *by a weight of* 50 *lbs.?*

Ans. 2,37717.

86. *Two planes* E M *and* M N, *having the same height, whose length are in the ratio of* 7 : 3, *meet at* M *above the horizontal line* E N; *and two weights* A *and* B, *connected by a string passing over a pulley at* M, *are in equilibrio on the planes, now the pressure upon* E M *is treble the pressure of* B *against* M N. *Required the height of these inclined planes.*

Ans. 2.

87. *A colossal statue* 18,08 *feet high, erected on the top of a column, subtends to an observer, whose eye is* 5,1 *feet above the level of the foot of the column, standing at a horizontal distance of* 125 *feet from it, the same angle, as a man of* 6 *feet standing at the foot of the column. What is the height of the column?*

Ans. 173,75.

88. *If a common glass bottle be corked and sunk in the sea* 400 *feet deep, what is the pressure upon the cork, if the mouth of the bottle be* ⅞ *of an inch diameter?*

Ans. ℔. 107½.

89. *A cylindrical vessel whose depth is three feet, was sunk into the ocean with the open end downwards, till the water rose* 31,5 *inches within the vessel. Determine how many feet under water the vessel was sunk, supposing the pressure of the atmosphere to be* 14¾ *lbs. on a square inch, and the compressibility of air to be uniform.*

Ans. 3262,5.

90. *If the specific gravity of mercury be* 14, *determine what ought to be the length of a water barometer inclined to the horizon at an angle of* 35° 45′ 30″, *the mercury standing at* 29,8 *inches in the tube.*

Ans. 59 *feet* & 5,4943 *inches.*

91. *Two spheres having* R *and* r *as radii, are just immersed in a fluid, what is the relative pressure on them?*

Ans. *The pressures are as* $R^3 : r^3$.

92. *Two ships being in the same degree of latitude but differing in longitudinal distance* 140 *nautical miles; after each had sailed* 2060 *sea miles due south, they are* 331,2681 *distance. From what latitude did the ships depart, supposing the earth to be a perfect sphere?*

Ans. *Latitude north* 66°.

By the third law of Kepler, that; The squares of the periodic times of any two planets are to each other, in the same proportion as the cubes of their mean distances from the sun:

93. *Find the distance of Mars from the Sun, the time of a revolution of the Earth being* 365,2563612 *days, and that of Mars* 686,9796458 *days, the distance of the Earth from the Sun* = 1.

Ans. 1,5236923.

94. *If a man weigh* 150 *lbs. on the earth, what would be his weight on the surface of the sun?*

Ans. 4185 ℔.

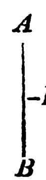
95. *If the true anamoly be* 30° 8′ 40″, *what are the excentric and mean anamolies?*

Ans. *Excentric* 32° 56′ 16′; *Mean* 35° 50′ 6″.

96. *Express the greatest equation of a planet* a, b, *and* c *being known.* *(See Ex. page 412.)*

$$Ans. \quad Tan. \frac{u}{2} = \sqrt{\frac{1 - \frac{\sqrt{a\,b}}{a+e}}{\frac{\sqrt{a\,b}}{a-e} - 1}}.$$

97. *From what height* A *above a given point* P, *must an elastic ball be let fall, so that after striking the plane at* B, *it may be reflected back again to* P *in the least time possible from the instant of dropping it?*



Ans. $A\,P = \frac{B\,P}{3}$.

98. *Determine the length* x *of an inclined plane whose height* h *is given, so that a known power acting by means of a thread passing over a fixed pulley, on a given weight, in a direction parallel to the plane, may draw it up in the least possible time.*

Ans. $x = 2\,h$.

99. *To divide a triangle* A B C *into two such parts that*

the one be to the whole triangle : : m : 1, *by means of a minimum straight line* M N = x.

$$Ans. \quad x = \frac{\sin. B}{\cos. \frac{B}{2}} \sqrt{\frac{a\,c}{m}}.$$

100. *If the angle* B *be bisected into two equal parts by a line* B E *then the required line* M N *cuts it vertically, therefore* B M = B N. *Determine also the position of* M N.

$$Ans. \quad B\,M = \sqrt{\frac{a\,c}{n}};\ and\ B\,E = \cos. \frac{B}{2} \sqrt{\frac{a\,c}{n}};\ E$$

being a point in M N.

101. *One extremity* C *of a beam* B C = a *feet long, is moveable about a fixed point in a vertical plane, and to the other extremity* B *a cord is fastened passing over a pulley* A *in a horizontal line with* C, *supporting a weight* P *equal to half the weight of the beam : required the position in which the beam will rest. Draw* B F *perpendicular to* A C. *Put* A C = b *and* A F = x.

$$Ans. \quad x = 0\ and\ x = b - \frac{a}{4\,b}\,(a \pm \sqrt{8\,b^2 + a^4}.$$

102. *Determine the length of a plane whose inclination is* 28° 40′ 55″, *so that a body let go at the top, may acquire a velocity of* 555⅓ *feet when it reaches the bottom.*

Ans. 10000 *feet.*

103. *A cylindrical lever* A C B *is bent and suspended at* C, *making an angle* ϕ, *about which point it is free to move in a verticle plane, and weights* P *and* P′ *are attached to its extremities, and* p, p′ *are the weights of the arms* A C = 2 a *and* C B = 2 b: *to find the position in which it will rest. Let* x *denote the angle the arm* A C *makes with a horizontal line.*

$$s. \quad Tan.\, x = \frac{(2\,P + p)\,a + (2\,P' + p')\,b\cos.\,\phi}{(2\,P' + p')\,b\sin.\,\phi}.$$

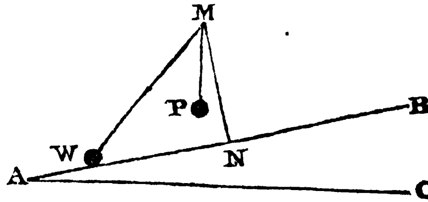
If the extremities of the lever carry no weights

$$tan.\, x = \frac{p\,a + p'\,b\cos.\,\phi}{p'\,b\sin.\,\phi}.$$

104. *An oblique cylinder stands on a horizontal plane to which it is inclined at an angle of* 60°, *its perpendicular height* 4 *feet, and diameter of the base* 3 *feet. Required the diameter of the greatest sphere, of the same material as the cylinder, that will hang suspended from its upper edge without upsetting the cylinder.*

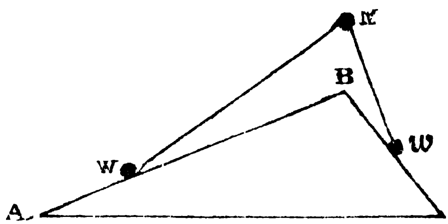
Ans. 2,006 *feet.*

105. *Given the position of the inclined plane* A B, *and of the pulley* M, *as also the weights of* W *and* P, *to determine whereabouts* W *must be placed that the equilibrium may be possible. Call the perpendicular* M N a, *which is given, because the position of* M *and of* A B *is given.*



$$Ans.\ W M = \frac{a}{\sin.\ B W M} = \frac{a P}{\sin.\ A \sqrt{P^2 - W^2 \sin.^2 A}}.$$

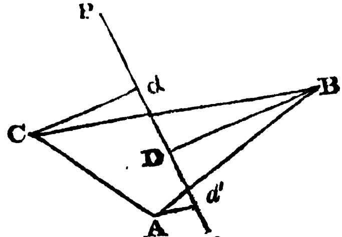
106. *Two weights* W, w, *attached to the extremity of a string, which passes over a fixed pulley* M, *mutually support each other on two planes* AB *and* A′B, *their inclination to the Horison being* A *and* A′ *degrees; to determine the relations between* W *and* w, *the tension of the string, and the pressures on the planes; let* e *and* e′ *denote the angles* M W B *and* M w B *of the inclined planes with the directions of the power.* R *and* R′ *the resistances on the planes* A B *and* A′ B; T *the tension.*



$$Ans.\ \frac{R}{W} = \frac{\cos.\ (A + e)}{\cos.\ e}; \frac{R'}{w} = \frac{\cos.\ (A' + e')}{\cos.\ e'}; \frac{T}{W} = \frac{\sin.\ A}{\cos.\ e}$$

$$\frac{T}{w} = \frac{\sin.}{\cos.\ e'} \quad \frac{W}{w} = \frac{\sin.\ A' \cos.\ e}{\sin.\ A \cos.\ e'}$$

107. *From a given point* P *without a known triangle* A B C, *to draw a straight line* P p *through the triangle, so that the perpendiculars* C d *and* A d′ *drawn on it from two of its angles, shall be equal to the perpendicular* B D *drawn in the opposite direction from the third angle. Finding*



first the distances P B, P C, P A, *and denoting them by* m. n, l, *determining also the angles* B P C $= \alpha$, *and* A P B $= \beta$, *and let* B P D $=$ x, *we have*

$$\text{Ans. Tan. } x = \frac{n \sin \alpha + l \sin \beta}{m + n \cos. \alpha + l \cos. \beta}.$$

108. *Suppose a colossal statue of* 30 *feet* 8 *inches were to be erected on the Ochterlony monument whose height is* 155 *feet, at what distance on the horizontal plane of the foot of the monument, must I station my theodolite* (4 *feet* 8 *inches*) *in order that the statue should subtend the same angle as would a man of* 5 *feet* 11 *inches at the distance from the theodolite of half the height of the monument?*

Ans. Either 86 *feet* 6 *inches nearly, or* 313 *feet* 9 *inches nearly.*

109. *Four persons* A, B, C, *and* D *have a number of rupees to divide. At first,* A *after making four equal divisions, finds one rupee over, which he gives to the poor and carries off one of the four parts. Then comes* B, *who also divides what* A *had left, into four equal parts, gives one rupee which he finds over, to the poor, and carries off one of the parts. After* B *comes* C *and after* C, D, *each acting the same way as* A *or* B *had done before. After which there remained a number of rupees which they divide equally amongst themselves without leaving any remainder. How much did each receive?*

Ans. A *received Rs.* 251; B, *Rs.* 203; C, *Rs.* 167, *and* D, *Rs.* 140.

110. *Let* A *and* B, *be two signals observed from a station* O, *and situate in the plane* O A B *which is not horizontal. Let* O Z *be the vertical to the point* O. O a b *the horizontal plane to which the angle* A O B *is required to be reduced. Put* δ *for* Z A *and* δ' *for* Z B, *the zenith distances of* A *and* B, *forming thus a spherical triangle* Z A B, *in which the three observed sides are known. Denoting the angle* A Z B, *by* Z *and by* S *the sum of the three sides* A B, δ, δ'.

$$\text{Ans. } \sin. \frac{Z}{2} = \sqrt{\frac{\sin.\left(\frac{s}{2} - \quad\right)\sin.\left(\frac{s}{2} - \delta'\right)}{\sin. \delta \sin.}}$$

$O = 62^\circ\ 4'\ 28''$

$\delta = 86^\circ\ 48'\ 40''$ *the reduced angle Z* $62^\circ\ 11'\ 10'',4$.

$\delta = 86^\circ\ 39'\ 54''$

111. *The edges of a rectangular parallelopiped are* E A, E B, E C, *and the diagonal* E M; *shew that the sum of the squares of the cosines of the diagonal with each of the three sides is equal to unity; viz.* $\cos.^2$ M E A + $\cos.^2$ M E B + $\cos.^2$ M E C = 1.

112. *Express the volume of a spherical sector; denoting by* h *the height of the segment on which it stands, and by* r *the radius of the sphere*

Ans. $\frac{2}{3} \pi r^2 h$

113. *Express the volume of a spherical segment.*

Ans. $\pi h^2 \left(r - \frac{h}{3} \right)$.

114. *Demonstrate that the area of a cycloid is equal to three times the generating circle.*

115. *Let* A *and* B *be two bodies of unequal weight, whose centres of gravity are united by the inflexible line* A B = *a Required the distance of* G *their common centre of gravity, from* A.

Ans. $A\,G = \frac{a\,B}{A + B}$.

116. *To find* G *the centre of gravity of a plane triangle.*

Ans. $\frac{2}{3}$ *of the line drawn from any angle to the middle of the opposite side counted from that angle.*

117. *If the distances* A G, B G, C G, *from the three angles of a triangle* A B C *be denoted by* m, n, p, *shew that* $a^2 + b^2 + c^2 = 3\,(m^2 + n^2 + p^2)$ *and* $m = \frac{1}{3} \sqrt{2 b^2 + 2 c^2 - 2 a^2}$ $n = \frac{1}{3} \sqrt{2 a^2 + 2 c^2 - 2 b^2}$, $p = \frac{1}{3} \sqrt{2 a^2 + 2 b^2 - 2 c^2}$.

118. *To find the centre of gravity of a pyramid or cone. Let* g *denote the centre of gravity of the base*, S *the angle at the vertex*, G *the centre of gravity required.*

Ans. $g\,G = \frac{g\,S}{4}$.

119. *If the pyramid is triangular,* A, B, C *denoting the angles of the base, and* S *that of the vertex,* G *its centre of gravity, demonstrate that*

$SG^2 + AG^2 + BG^2 + CG^2 = \frac{1}{4}\,(SA^2 + SB^2 + SC^2 + AB^2 + AC^2 + BC^2)$

If the sides of the base are equal; $S\,G^2 = \frac{1}{16}\,(S\,A^2 + S\,B^2 + S\,C^2 - A\,B^2)$

120. *If all the edges of the triangular pyramid are equal, shew that in the regular tetrahedron.* (G *denoting the centre of gravity, and* A *an angle at the base)*

$$S\,G = \frac{S\,A}{4}\sqrt{6}.$$

121. *Find the centre of gravity of a frustrum of a cone or pyramid. Put* R *and* r *the radii of the greater and less end, and* h *for the height of the frustrum.*

Ans. The distance on the axis from the centre of the lesser base $= \frac{h}{4} \cdot \frac{3\,R^2 + 2\,R\,r + r^2}{R^2 + R\,r + r^2}$.

122. *Shew that the centre of gravity of a paraboloïd from the vertex is* $\frac{2}{3}$ *of the axis.*

123. *Shew that the centre of gravity of the frustrum of a paraboloïd, is at the distance on the axis from the centre of the lesser base* $= \frac{h}{3} \cdot \frac{2\,R^2 + r^2}{R^2 + r^2}$.

124. *To find the centre of gravity of a circular segment* A N B; *Let* N *be the middle point of the arc* A N B; C *the centre and* N P C *the radius* = r *of the circle,* G *the centre of gravity, and put* C P = a.

$$\textit{Ans.}\quad C\,G = \frac{2\,(r^2 - a^2)^{\frac{3}{2}}}{3\ \textit{area segment}}.$$

125. *Inscribe the greatest ellipse in a given isosceles triangle. Put* h *for the height* B D, *and* 2 b *for* A C *the base of the triangle*; 2 x *the greater axis of the ellipse* D a.

$$\textit{Ans.}\quad x = \frac{h}{3}.$$

126. *Inscribe the greatest parabola in a given isosceles triangle, x denoting the abscissa.*

$$\textit{Ans.}\quad x = \frac{h}{2}.$$

127. *Required the least triangle* ABC *which can be described about a given quadrant of which the radius is* = r.

Ans. A right angled isosceles triangle.

128. *Within a given parabola, inscribe the greatest parabola, the vertex of the latter being at the bisection of the base of the former. Put* a *for the abscissa of the given parabola, and* p *for its parameter.*

$$\textit{Ans.}\quad x = \frac{a}{3}.$$

129. *The corner of a leaf is turned back, so as just to reach the other edge of the page: find when the length of the crease is a minimum. Let* A C = b *be the base of the leaf;* P, *a point in* A C, P B = u = *the crease,* A P = x.

$$\textit{Ans.} \quad u = x\sqrt{\frac{2x}{2x - b}} \qquad x = \frac{3b}{4}.$$

130. *Let* u = (m x + n). (n y + m) *be a maximum, and let* a^{mx}. b^{ny} = c; *find* x.

$$\textit{Ans.} \quad x = \frac{\log.\left(\frac{cb^m}{a^n}\right)}{\log.\,(a^{2m})}.$$

131. *Inscribe the greatest rectangle in a given triangle. Put* h *the height, and* b *the base of the triangle;* M N *for the base of the rectangle,* x *for its height.*

$$\textit{Ans.} \quad x = \frac{h}{2},\ M\,N = \frac{b\,x}{h}.$$

132. *Inscribe the greatest rectangle in a semicircle. Be* r = *radius of the circle;* 2 x *the base of the rectangle,* s = *area of the maximum rectangle.*

$$\textit{Ans.} \quad s = r^2\sqrt{2},\ x = \frac{r}{\sqrt{2}}.$$

133. *Inscribe the greatest rectangle in a given parabola. Put* a *the axis,* x = *the part of the axis counted from the vertex of the parabola to the point where it is cut by the rectangle.*

$$\textit{Ans.} \quad x = \frac{a}{3}\ ;\ \textit{height of the rectangle} = \frac{2a}{3}.$$

134. *Inscribe in a given segment of a circle the greatest rectangle. Put* r = *radius of the circle,* a *the height of the segment, its base will then be expressed by* $2\sqrt{2rx - x^2}$.

$$\textit{Area} = \frac{3r - a \pm \sqrt{9r^2 - 2ar + a^2}}{4}.$$

135. *Inscribe the greatest parallelopipedon within a given ellipsoid. Let* 2 x, 2 y, 2 z *be the edges* ∴ u = *the maximum parallelopipedon;* 2 a, 2 b, 2 c *the principal diameters of the ellipsoid;* u = 8 x y z *is a maximum.*

$$\textit{Ans.} \quad x = \frac{a}{\sqrt{3}};\ y = \frac{b}{\sqrt{3}};\ z = \frac{c}{\sqrt{3}} \therefore u = \frac{8abc}{3\sqrt{3}}.$$

136. *Express* x *the radius of a ball, which, being placed in a conical glass full of water, shall expel the most water possible from the glass; the depth being* b, 2 a *the diameter of the glass, and for* $\sqrt{a^2 + b^2}$ *put* c.

$$Ans. \quad x = \frac{a\,b\,c}{(c - a)\,(2\,a + c)}.$$

137. *A homogeneous lever* A B *of the second kind is equally thick throughout, the fulcrum being at* A; *it is required to determine what must be the length of the arm* A B, *that a given weight* W, *suspended at* C *at a distance of* p *inches from* A, *may be supported by the least power* P *possible acting at the extremity* B, *taking into account the weight of the lever itself; the lever being homogeneous and equally thick throughout, any portion of its length may be taken to represent the weight of that portion; hence, calling the length* A B *of the lever* x, *its weight will be* x, *acting at* G, *its middle point.*

$$Ans. \quad x = \sqrt{2\,W\,p}.$$

138. *Cut the greatest ellipse from a given cone.* A B C *the cone,* A P *the elliptic section;* B D = α *the axis of the cone;* A D = β; P N = y *the perpendicular on the diameter* A C *of the base of the cone;* D N = x. A P = 2 a *the axis major of the cone, and the axis minor* = 2 b.

$$Ans. \quad x = \frac{2\,\beta\,(\alpha \qquad \beta^2) \pm \beta\sqrt{\alpha^4 - 14\,\alpha^2\,\beta^4 + \beta^4}}{3\,(\alpha^4 + \beta^4)}.$$

The limit of possibility is when the radical disappears.

139. *Cut the greatest parabola from a given cone. Let* A C = b *denote the diameter of the base of the cone;* P G *the axis of the parabola;* 2 y *its base, and* C G = x.

$$Ans. \quad x = \frac{3\,b}{4}; \; y = \frac{b}{4}\sqrt{3}.$$

140. *Cut the greatest cylinder out of a given cone. Let* b *be the diameter of the base of the cone,* h *the height.*

Ans. Diameter of the base of the max. cylinder $\frac{2\,h}{3}$; *height of cylinder* $= \frac{h}{3}$.

141. *The volume of a cone being given, find its form when its total surface* u *is a maximum. Let* x *denote the height, and* y *the radius of the base.*

$$x = 2\,a; \; y = \frac{a}{\sqrt{2}}; \; \frac{x^2}{y^2} = 8;$$

$$Ans. \quad u = 2\,\pi\,a^2.$$

142. *Divide a line into two such parts, that their product multiplied by the difference of their squares shall be a maximum. Put* 2 a *for the line* ∴ a + x *and* a — x *the two parts.*

Ans. $x = \frac{a}{\sqrt{3}}$.

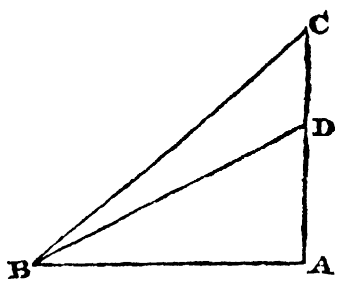
143. $u = \frac{\log. x}{x}$, *find* x *that* u *may be a maximum.*

Ans. $u = \frac{1}{e}$.

144. *Find that fraction which exceeds its second power by the greatest possible number. Be* x *the fraction.*

Ans. $x = \frac{1}{2}$.

145. *Determine the distance of* B *from* A, *that* ∠ D B C *may be a Maximum. Put* A D = a, A C = b ; A B = x ; ∠ D B C = ϕ.



Ans. $x = \sqrt{a\,b}$ = *a tan. to a circle circumscribing the* △ B C D.

146. *Determine* x *that* u *may be a* $\left\{\begin{matrix} maximum \\ minimum \end{matrix}\right\}$ *in the equation* $u^3 - 3\,a\,u\,x + x^3 = 0$.

$x = a\sqrt[3]{2}$, *gives* $u = a\sqrt[3]{4}$ *a maximum.*

$x = 0$, *gives* $u = 0$ *a minimum.*

147. *Bisect a triangle* A B C *by the shortest* [illegible] M N. *Call* C M = z, C N = y ; M N = x.

Ans. $x = \sqrt{(b - a + c)\ (b + a - c)}$ [illegible]

148. *Describe about a given circle whose radius* O D = r, *the least isosceles triangle.* A B C *the triangle* ; A B *touching the circle at* D ; O *the centre of the circle,* B O = x ;

B D = $\sqrt{x^2 - r^2}$. [illegible] $r^2\,3\sqrt{3}$.

149. *Find the greatest area that can be included by four given straight lines,* a, b, c, d.

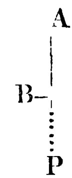
Ans. *The quadrilateral inscribable in a circle*

$$= \sqrt{(P - a)(P - b)(P - c)(P - d)}.$$

150. *Demonstrate that the maximum area of a triangle of an equal perimeter, is that which is equilateral.*

151. *Demonstrate that of all parallelópipeds of the same surface and height, the cube has the greatest volume.*

152. *It is required to find a point* P *on the prolongation of the given line* A B = a, *so that the square of the distance* A P (x), *being divided by* B P, *the quotient be a* $\begin{cases} \textit{maximum.} \\ \textit{minimum.} \end{cases}$



Ans. *For the max.* $x = a$; *for the min.* $x = 2a$.

153. *Through a given point* P *within an angle* A B C, *to draw a straight line* M N, *so that it may form with the two sides of the angle* A B C *the least possible triangle. From* P *draw a line parallel to one of the sides of the angle till it meets the other side, at* D ; *call* B D = a ; *and* x *the base* B D M.

Ans. $x = 2a$.

154. *To divide a quantity* a *into three parts* x, y, a — x — y ; *such that the product* $x^m y^n (a - x - y)^p$ *be a maximum.*

$$\textit{Ans.} \quad x = \frac{m a}{m + n + p}; \; y = \frac{n a}{m + n + p};$$

$$a - x - y = \frac{p a}{m + n + p}.$$

155. *A given weight* W′, *is to draw another given weight* W *upon an inclined plane of a given height* h ; *required the length* l *of the plane that the time of ascent may be a minimum. The inclination of the plane being represented by* A, *we have*

$$\sin. A = \frac{h}{l}.$$

$$\textit{Ans.} \quad l = \frac{2 W h}{W'}.$$

156. *To de*[illegible]*gle the wind must strike against the sails of a* [illegible] *to put it* [illegible]*tion, may be a maximum.*

Ans. 54° 44′ 8″.2.

157. *Given* A B = C D = b, *the length of a man's foot, and* A C = a, *the distance of his heels; to find the position of his feet when he stands firmest. Draw* A E *perpendicular from* A, *one of his heels on* B D *the line joining the front part of the feet. Call* B E = x.

$$Ans.\quad x = \frac{-a + \sqrt{a^2 + 8b^2}}{4}.$$

If the heels touch, viz. a = 0, *then the feet are at right angles to each other.*

158. *A wheel is connected with a cylinder or axle, having its centre in the axis of the cylinder about which the whole turns. From the circumference of the wheel whose radius* = R, *is suspended a weight* P, *and from the axle whose radius* = r, *is suspended a weight* W. *What must be the comparative length of* R *and* r, *that the effect may be the greatest, viz. in which case will the velocity of* W *be the greatest?*

$$Ans.\quad r = R\sqrt{\frac{P}{W}}.$$

If the moving force P = 10; *the weight* = 100000 *lbs. then*

$$r = R\sqrt{\frac{10}{100000}} = \frac{R}{100};$$ *viz. that the radius of the cylinder must be the* 100*th part of that of the wheel.*

159. *Determine the length of the shortest line* M N *to pass through a given point* P *within a right angle* A B C. *Put* a *for the perpendicular distance from the given point to the line* B C; *and* b *for. . . . . do. . [illegible] . . . . do. . . . .* A C.

$$Ans.\quad M N = a\sqrt{1 + \frac{b^{\frac{2}{3}}}{a^{\frac{2}{3}}}} + b\sqrt{1 + \frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}}.$$

160. *It is required to div[illegible] a given arc* A *int[illegible] parts, such [illegible] mth po[illegible]e sine of one part, m[illegible] by the [illegible] of th[illegible] the other part, shall [illegible] mum. [illegible] X and* Y *denote the two parts,* x *and* y *thei[illegible]s.*

$$Ans.\quad Sin.\,(X - Y) = \frac{m - n}{m\ldots}\ [illegible]\ A.$$

161. *To determine the situation* [illegible] *us* [illegible] *lation to the Earth, when she has the* [illegible] *illiancy? Put* m *the distan*[illegible] *Earth fr*[illegible]
n *do.* [illegible] *Venus* [illegible]

$$Ans.\quad [illegible]\ m^2 + n^2.$$

If the Earth is in her apogee and Venus in her perigee, ...	*Angle of elongation of Venus*	39° 6′.
If the Earth and Venus are at their mean distances from the sun, ...		39° 43′.
If the Earth is in her perigee and Venus in her apogee, ...		40° 22′.

162. *It is required to divide a given number* a, *into two such parts, that the* m*th power of one part multiplied by the* n*th power of the other be a maximum. Let one part be denoted by* x, *and* $x^m y^n = max.$

$$Ans.\quad x = \frac{m\,a}{m+n};\ y = \frac{n\,a}{m+n}.$$

163. *Given* A D *a parabolic curve whose vertex is* A, *and axis* A Z *parallel to the horizon; to find that point* P *in the curve, where a body falling along the curve from* A *would strike an horizontal plane with the greatest force. Draw the tangent* T P *meeting the axis produced, from any point in the curve* P, *draw the ordinate* P N = y, *call* A N = x ; *and the parameter* = p.

Ans. $x = \frac{p}{4}$; *therefore N is the focus of the parabola.*

164. *Let* A *be the vertex of a parabola* A D.D′, *whose axis* A Z *is perpendicular to the* [illegible], C *any given point in* A Z ; *to find* D C *the line of qu*[illegible]*escent from the curve to* C ; *and* C D′ *the line of quickest descent from* C *to the curve. Put* A C = a *and* [illegible] = p a *the corresponding ordinate,* p = *parameter,* A B = x, *and* A B′ = x′ *the abscissa corresponding to the ordinates* D B *and* [illegible]*icular to the axis* [illegible]

[illegible] *Ans.* [illegible] *na.*

[illegible] B *be a give*[illegible] *to w*[illegible]*defi-* [illegible] *is drawn* [illegible] N *take*[illegible] *points* D [illegible] *which draw* D E *and* C F [illegible]*dicular* [illegible] A D *and* B C. *Suppo*[illegible] *move* [illegible] B *with given veloc*[illegible]*respec-tive* [illegible] *time t, to find* [illegible] *and* D. [illegible] E = C F = d. [illegible] d [illegible] n

V *and* u *must be so limited that*

$$\frac{V\,d}{\sqrt{v^2 - V^2}} + \frac{v\,d}{\sqrt{v^2 - u^2}} = \text{or} \angle a.$$

166. *Let there be two lights at* M *and* N, *in the ratio* m : 1 ; a *being the distance between them to find the point* C *between them where the least light is received; also the locus of all the points where a proportional quantity is received.*

Ans. $N\,C = \dfrac{a}{m^{\frac{1}{3}} + 1}$. *Having determined* C *take* C O : M C : : N C : M C — N C : *then with the centre* O *and radius* O C *describe a circle, and its circumference will be the locus required.*

167. *Given the base* A C = b *to find the perpendicular* B A = x *such that a body falling from* B *to* A, *and then describing* A C *with the velocity acquired, the time through* B A *and* A C *may be the least possible.*

Ans. $x = \dfrac{b}{2}$.

168. *Given the base* A C = b *of an inclined plane* B C, *to find the altitude* A B = x, *such that the horizontal velocity of a body at* C *after descending down* B C, *may be the greatest possible.*

Ans. $x = b$.

169. *Given t*[illegible] *proportion between* y *the* [illegible] *the time of its vibr*[illegible] *nt of suspension* [illegible]

Tables for the Resolution of a Plane Triangle A B C, from three given Parts.

N. B. The sides are denoted by corresponding Italic small characters.

$a =$

1, $\dfrac{b \sin. A}{\sin. B}$

2, $\dfrac{c \sin. A}{\sin. C}$

3, $\dfrac{b}{\cos. C + \sin. C \cot. A}$

4, $\dfrac{c}{\cos. B + \sin. B \cot. A}$

5, $b \cos. C + b \sin. C \cot. B$

6, $C \cos. B + c \sin. B \cot. C$

7, $\sqrt{b^2 + c^2 - 2\,b\,c \cos. A}$

8, $b \cos. C \pm \sqrt{c^2 - b^2 \sin.^2 C}$

9, $c \cos. B \pm \sqrt{b^2 - c^2 \sin. [illegible]}$

[illegible]

1, $\dfrac{a \sin. C}{c}$

2, $\dfrac{a \sin. B}{b}$

3, Sin. (B [illegible]

$\text{Cos. A} =$

1, $\dfrac{\pm \sqrt{c^2 - a^2 \sin.^2 C}}{c}$

2, $\dfrac{\pm \sqrt{b^2 - a^2 \sin.^2 B}}{b}$

3, $-\cos. (B + C)$

4, $\text{Sin. B} \sin. C - \cos. B \cos. C$

5, $\dfrac{b - a \cos. C}{\sqrt{a^2 + b^2 - 2\,a\,b \cos. C}}$

6, $\dfrac{c - a \cos. B}{\sqrt{a^2 + c^2 - 2\,a\,c \cos. B}}$

7, $\dfrac{b^2 + c^2 - a^2}{2\,b\,c}$

8, $\dfrac{b \sin.^2 C \mp \cos. C \sqrt{c^2 - b^2 \sin.^2 C}}{c}$

9, $\dfrac{c \sin.^2 B \mp \cos. B \sqrt{b^2 - c^2 \sin.^2 B}}{b}$

[illegible]

1, $\dfrac{a \sin. C}{\pm \sqrt{c^2 - a^2 \sin.^2 C}}$

2, $\dfrac{a \sin. B}{\sqrt{b^2 - a^2 \sin.^2 B}}$

[illegible] (B + C)

$\dfrac{[illegible] \tan. C}{[illegible] C - 1}$

Table of the principal formulæ of a spheric triangle A B C.

$$s = \frac{a+b+c}{2};\quad S = \frac{A+B+C}{2}.$$

$$Tan.\ \frac{A}{2} = \sqrt{\frac{sin.\ (s-b)\ sin.\ (s-c)}{sin.\ s\ sin.\ (s-a)}}$$

$$= \frac{cos.\ \frac{1}{2}\ (b-c)}{cos.\ \frac{1}{2}\ (b+c)}\ cot.\ \tfrac{1}{2}\ (B+C).$$

$$= \frac{sin.\ \frac{1}{2}\ (b-c)}{sin.\ \frac{1}{2}\ (b+c)}\ cot.\ \tfrac{1}{2}\ (B-C).$$

$$Sin.\ \frac{A}{2} = \sqrt{\frac{sin.\ (s-b)\ sin.\ (s-c)}{sin.\ b\ sin.\ c}};$$

$$Cos.\ \frac{A}{2} = \sqrt{\frac{sin.\ s\ sin.\ (s-a)}{sin.\ b\ sin.\ c}}.$$

$$Tan.\ \frac{a}{2} = \sqrt{\frac{-\ cos.\ S\ cos.\ (S-A)}{cos.\ (S-B)\ cos.\ (S-C)}}$$

$$= \frac{cos.\ \frac{1}{2}\ (B+C)}{cos.\ \frac{1}{2}\ (B-C)}\ tan.\ \tfrac{1}{2}\ (b+c)$$

$$= \frac{sin.\ \frac{1}{2}\ (B+C)}{sin.\ \frac{1}{2}\ (B-C)}\ tan.\ \tfrac{1}{2}\ (b-c).$$

$$Sin.\ \frac{a}{2} = \sqrt{\frac{-\ cos.\ S\ cos.\ (S-A)}{sin.\ B\ sin\ C}};$$

$$Cos.\ \frac{a}{2} = \sqrt{\frac{cos.\ (S-B)\ cos.\ (S-C)}{sin.\ B\ sin.\ C}}.$$

$$Sin.\ A = \frac{sin.\ a\ \ldots}{\ldots}$$

$$\text{Cos. [illegible]} = \frac{\cos. a + \cos. b + \cos. c + 1}{4 \cos. \frac{a}{2} \cos. \frac{b}{2} \cos. \frac{c}{2}}.$$

$$= \frac{\cos.^2 \frac{a}{2} + \cos.^2 \frac{b}{2} + \cos.^2 \frac{c}{2} - 1}{2 \cos. \frac{a}{2} \cos. \frac{b}{2} \cos. \frac{c}{2}}.$$

$$\text{Tan.} \frac{\Sigma}{4} = \sqrt{\left[\tan. \frac{s}{2} \tan. \left(\frac{s-a}{2}\right) \tan. \left(\frac{s-b}{2}\right) \tan. \left(\frac{s-c}{2}\right)\right]}.$$

N. B. These formulæ are much simplified when the proposed triangle has one of its angles a right angle. Supposing $\angle C = 90°$, then sin. $C = 1$, cos. $C = 0$.